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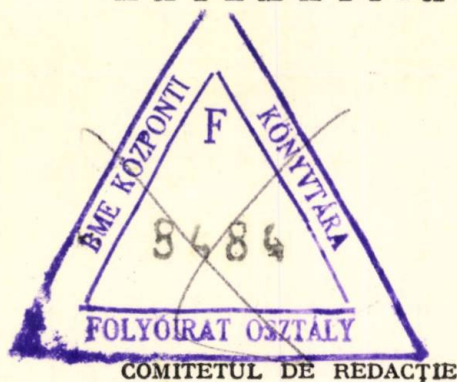
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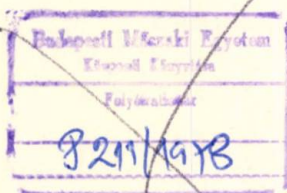
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Redactor responsabil : Prof. dr. doc. *Cabrita Andrelan*
Redactor responsabil adjunct : Conf. dr. doc. *P. P. Teodorescu*
Membri : Dr. doc. *Aurel Cornea*
 Prof. dr. doc. *Romulus Cristescu*
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REDACȚIA
TIPOGRAFIA UNIVERSITĂȚII BUCUREȘTI
Șoseaua Panduri, nr. 90
Telefon : 314000/132

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SUMAR • SOMMAIRE •

БАЙНОВ Д.Д., МИЛУШЕБА С.Д., Применение метода усреднения к одной двухточечной краевой задаче для систем интегро-дифференциальных уравнений типа Фредгольма неразрешенных относительно производной	3
BUIUMA., Sur les ideaux lissés	15
DINESCU C., SĂVULESCU B., New Results in the Flow Transport Problem	21
КОНСТАНТИНОВ М. М., БАЙНОВ Д. Д. Краевая задача с параметром для дифференциального уравнения с отклоняющимся аргументом	37
MARCU D., A Method for Finding the Kernel of a Digraph	41
MICU N., NIȚULESCU F., Un espace isométrique à l'espace Paul Lévy des fonctions de répartition	45
NICOLESCU L., Sur l'algèbre de déformation associée à une variété de Riemann	51
PACHPATTE B. G., Some Problems in Nonlinear Differential Equations with Integral Perturbations	57
RĂDULESCU M., RĂDULESCU S., Irreducible Polynomials	65
SHARMA B. L., On the Generalised Function of two Variables (III)	79
TEODORESCU P. P., Sur une solution en tensions de l'élastodynamique linéaire	91
UDRIȘTE C., Properties of a Functional Defined on the Paths Space of a Riemannian Manifold	97
VERMA R. U., Integral Equations Involving the G-Function as Kernel	107
ZAMFIR M., Initial Algebra Semantics for Knuth Systems with Inherited Attributes	111
VIAȚA FACULTĂȚII: Profesorul Gheorghe Galbură la 60 de ani	123

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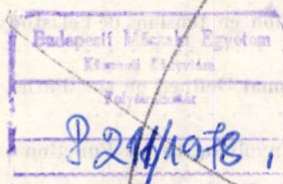
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ПРИМЕНЕНИЕ МЕТОДА УСРЕДНЕНИЯ К ОДНОЙ ДВУХТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ СИСТЕМ ИНТЕГРО — ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТИПА ФРЕДГОЛЬМА НЕРАЗРЕШЕННЫХ ОТНОСИТЕЛЬНО ПРОИЗВОДНОЙ

Д.Д. БАЙНОВ, С.Д. МИЛУЩЕВА

В работах [1], [2] был обоснован метод усреднения для решения двухточечной краевой задачи для систем обыкновенных дифференциальных уравнений. В настоящей работе этот метод применяется для решения нелинейной двухточечной краевой задачи для систем интегро-дифференциальных уравнений типа Фредгольма неразрешенных относительно производной.

Надо отметить, что метод усреднения для задачи Коши для систем интегро — дифференциальных уравнений был обоснован в работах [3] — [6].

Пусть для системы

$$\dot{x}(t) = \varepsilon X(t, x(t), y(t), \dot{x}(t), \dot{y}(t), \int_0^1 \varphi(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s)) ds), \quad (1)$$

$$\dot{y}(t) = Y(t, x(t), y(t), \dot{x}(t), \dot{y}(t), \int_0^1 \varphi_1(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s)) ds),$$

где

$$x = (x^{(1)}, \dots, x^{(n)}) \quad y = (y^{(1)}, \dots, y^{(m)}), \quad X = (X^{(1)}, \dots, X^{(n)})$$

$$Y = (Y^{(1)}, \dots, Y^{(m)}), \quad \varphi = (\varphi^{(1)}, \dots, \varphi^{(q)}), \quad \varphi_1 = (\varphi_1^{(1)}, \dots, \varphi_1^{(r)})$$

а $\varepsilon > 0$ — малый параметр, задано краевое условие

$$x(0) = x^0, \quad R[\lambda, y(0), y(T)] = 0, \quad (2)$$

$$y = (\lambda^{(1)}, \dots, \lambda^{(m)}), \quad R = (R^{(1)}, \dots, R^{(m)}), \quad \lambda \in \Lambda \subseteq R_m,$$

$$T = L\varepsilon^{-1}, \quad L = \text{const} > 0$$

Наряду с системой (1) рассмотрим вырожденную ($\varepsilon = 0$) по отношению к ней систему

$$x = \text{const.}, \quad \dot{x} = \text{const.},$$

$$\dot{y}(t) = Y(t, x, y(t), \dot{x}, \dot{y}(t), \int_0^1 \varphi_1(t, s, x, y(s), \dot{x}, \dot{y}(s)) ds), \quad (3)$$

с краевым условием

$$R[\lambda, y(0), y(T)] = 0 \quad (4)$$

Пусть вдоль интегральных кривых $y = \psi(t, x, \dot{x}, \lambda, T)$ краевой задачи (3), (4), где λ рассматривается как векторный параметр, существует независящее от параметра λ среднее значение

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t, x, \psi(t, x, \dot{x}, \lambda, T), \dot{x}, \frac{\partial}{\partial t} \psi(t, x, \dot{x}, \lambda, T), \varphi(t, s, x, \psi(s, x, \dot{x}, \lambda, T), \dot{x}, \frac{\partial}{\partial s} \psi(s, x, \dot{x}, \lambda, T))) ds dt = \bar{X}(x, \dot{x}). \quad (5)$$

Тогда усредненным уравнением первого приближения для медленных переменных $x(t)$ системы (1) будем называть уравнением

$$\dot{\xi}(t) = \epsilon X(\xi(t), \dot{\xi}(t)) \quad (6)$$

с начальным условием

$$\xi(0) = x^0 \quad (7)$$

Отметим, что если $x = (x^{(1)}, \dots, x^{(n)})$ и $A = (a_{ij})_{n,m}$, то по определению

$$\|x\| = \left[\sum_{i=1}^n (x^{(i)})^2 \right]^{\frac{1}{2}}, \quad \|A\| = \left[\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right]^{\frac{1}{2}}$$

Докажем теорему о близости компонент $x(t)$ решения $\{x(t), y(t)\}$ краевой задачи (1), (2) и решения задачи Коши (6), (7).

ТЕОРЕМА 1. Пусть:

1. Функции $\varphi(t, s, x, y, u, v)$ и $\varphi_1(t, s, x, y, u, v)$ определены, непрерывны и равномерно ограничены вместе со своими частными производными по t для всех $t \geq 0, s \geq 0, (x, y, u, v) \in G$, где G некоторая открытая область $2(n+m)$ -мерного пространства переменных x, y, u, v .

Функция $X(t, x, y, u, v, z)$ определена, непрерывна и равномерно ограничена вместе со своими частными производными первого порядка для всех $t \geq 0$ и $(x, y, u, v, z) \in G \times G_1$, где G_1 — некоторая открытая область пространства R_q .

Функция $Y(t, x, y, u, v, w)$ определена, непрерывна и равномерно ограничена вместе со своими частными производными первого порядка для всех $t \geq 0$ и $(x, y, u, v, w) \in G = G \times G_2$ где G_2 — некоторая открытая область пространства R_7 .

2. Существуют равномерно ограниченные в соответствующих проекциях области $G^* = \{t \geq 0, s \geq 0, G \times G_1 \times G_2\}$ обратные матрицы

$$\left(I_1 - \varepsilon \frac{\partial X}{\partial u}\right)^{-1}, \left(I_2 - \frac{\partial Y}{\partial v}\right)^{-1}, \left[I_2 - \frac{\partial Y}{\partial v} - \varepsilon \frac{\partial Y}{\partial u} \left(I_1 - \varepsilon \frac{\partial X}{\partial u}\right)^{-1} \frac{\partial X}{\partial v}\right]^{-1},$$

где I_1 и I_2 — соответственно n и m — мерные единичные матрицы.

3. Через каждую точку области $G^*(t, x, y, u)$ ($G^*(t, x, y, u)$ — проекция области G^* в пространстве переменных t, x, y, u) проходит единственная интегральная кривая краевой задачи (3), (4), соответствующая некоторому значению параметра λ , причем это решение определено для всех значений, $t \in [0, \infty)$ и при этих значениях t целиком лежит в $G^*(x, y, u)$.

Пусть указанные решения краевой задачи (3), (4) задаются в форме

$$y = \psi(t, x, \dot{x}, \lambda, T), \quad (8)$$

где ψ — m — мерная, непрерывная по совокупности всех переменных вектор — функция,

4. Для всех $(x, u, \lambda) \in G^*(x, u) \times \Lambda$ существует независимый от параметра λ предел

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t, x, \psi(t, x, u, \lambda, T), u, \frac{\partial}{\partial t} \psi(t, x, u, \lambda, T), \quad (9)$$

$$\int_0^1 \varphi(t, s, x, \psi(s, x, u, \lambda, T), u, \frac{\partial}{\partial s} \psi(s, x, u, \lambda, T) ds) dt = \bar{X}(x, u),$$

причем предельный переход в (9) происходит равномерно относительно совокупности $(x, u, \lambda) \in G^*(x, u) \times \Lambda$. Среднее значение $\bar{X}(x, u)$ — определена, непрерывна и равномерно ограничена функция для всех $(x, u) \in G^*(x, u)$ и удовлетворяет в $G^*(x, u)$ неравенством

$$\|\bar{X}(x, u) - \bar{X}(x', u')\| \leq c (\|x - x'\| + \|u - u'\|),$$

где c — положительная постоянная.

5. Краевая задача (1), (2) имеет единственное решение $\{x(t), y(t)\}$ удовлетворяющее условием

$$x(0) = x^0, R[\lambda, y(0), y(T)] = 0.$$

где под λ подразумевается некоторое фиксированное значение параметра λ . Решение $\{x(t), y(t)\}$ определено на отрезке $[0, T]$ при всех значениях $\varepsilon \in (0, \varepsilon^*]$ и не выходит из некоторой открытой подобласти $G^{*1}(x, y) \subset G^*(x, y)$ лежащей целиком внутри $G^*(x, y)$ вместе со своей границей.

6. Решение усредненного уравнения

$$\dot{\xi}(t) = \varepsilon \bar{X}(\xi(t), \xi(t)) \quad (10)$$

удовлетворяющее начальному условию $\xi(0) = x^0$ существует, однозначно определено при $t \geq 0$ для всех значениях $\varepsilon \in (0, \varepsilon^*]$ и кривая $\{\xi(t), \xi(t)\}$ соответствующая этому решению на интервале $0 \leq t \leq \infty$ не выходит из области $G^*(x, u)$.

7. Уравнение в частных производных

$$\begin{aligned} \frac{\partial W}{\partial t} + \frac{\partial W}{\partial y} Y + \frac{\partial W}{\partial v} \left(I_2 - \frac{\partial Y}{\partial v} \right)^{-1} \left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} Y + \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial \varphi_1}{\partial t} ds \right) = \\ = X(t, x, y, u, v, z) - \bar{X}(x, u), \end{aligned} \quad (11)$$

$$W|_{t=0} = 0, \quad x = \text{const.}, \quad u = \text{const.},$$

обладает некоторым решением $W = W(t, x, y, u, v)$, которое определено и равномерно ограничено вместе со своими частными производными первого порядка во всей области G^* .

Тогда если $\{x(t), y(t)\}$ решение краевой задачи (1), (2), а $\xi(t)$ — решение задачи Коши (6), (7), то для любого сколь угодно малого $\delta > 0$ можно указать такое $\varepsilon^0 > 0$, что при $0 < \varepsilon \leq \varepsilon^0$ на отрезке $0 \leq t \leq T$ будет выполняться неравенство

$$\|x(t) - \xi(t)\| < \delta$$

Доказательство. Обозначим через $\eta(t)$ выражение

$$\eta(t) = x(t) - \varepsilon W(t, x(t), y(t), \dot{x}(t), \dot{y}(t)), \quad (12)$$

где $\{x(t), y(t)\}$ — решение краевой задачи (1), (2), а W — непрерывно дифференцируемое решение (11),

Оценим разность $\dot{\eta}(t) - \dot{\xi}(t)$ где $\xi(t)$ — решение усредненного уравнения (10) с начальным условием $\xi(0) = x^0$

Имеем

$$\dot{\eta}(t) - \dot{\xi}(t) = \dot{x}(t) - \varepsilon \left\{ \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} \dot{x}(t) + \frac{\partial W}{\partial y} \dot{y}(t) + \frac{\partial W}{\partial \dot{x}} \ddot{x}(t) + \right. \\ \left. + \frac{\partial W}{\partial \dot{y}} \ddot{y}(t) + \bar{X}(\xi(t), \dot{\xi}(t)) \right\}.$$

Для определения $\ddot{x}(t)$ и $\ddot{y}(t)$ продифференцируем систему (1):

$$\ddot{x}(t) = \varepsilon \left\{ \frac{\partial X}{\partial t} + \frac{\partial X}{\partial x} \dot{x}(t) + \frac{\partial X}{\partial y} \dot{y}(t) + \frac{\partial X}{\partial \dot{x}} \ddot{x}(t) + \frac{\partial X}{\partial \dot{y}} \ddot{y}(t) + \right. \\ \left. + \frac{\partial X}{\partial z} \int_0^1 \frac{\partial}{\partial t} \varphi(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s)) ds \right\},$$

$$\ddot{y}(t) = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial x} \dot{x}(t) + \frac{\partial Y}{\partial y} \dot{y}(t) + \frac{\partial Y}{\partial \dot{x}} \ddot{x}(t) + \frac{\partial Y}{\partial \dot{y}} \ddot{y}(t) + \\ + \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial}{\partial t} \varphi_1(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s)) ds.$$

Отсюда

$$\left(I_1 - \varepsilon \frac{\partial X}{\partial \dot{x}} \right) \ddot{x}(t) - \varepsilon \frac{\partial X}{\partial \dot{y}} \ddot{y}(t) = \varepsilon \left\{ \frac{\partial X}{\partial t} + \frac{\partial X}{\partial y} \dot{y} + \right. \\ \left. + \frac{\partial X}{\partial z} \int_0^1 \frac{\partial \varphi}{\partial t} ds \right\} + \varepsilon^2 \frac{\partial X}{\partial x} \dot{x}, \\ - \frac{\partial Y}{\partial \dot{x}} \ddot{x}(t) + \left(I_2 - \frac{\partial Y}{\partial \dot{y}} \right) \ddot{y}(t) = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} \dot{y} - \\ - \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial \varphi_1}{\partial t} ds + \varepsilon \frac{\partial Y}{\partial x} \dot{x}. \quad (14)$$

Решение системы (14) относительно $\ddot{x}(t)$ и $\ddot{y}(t)$ представляем в виде

$$\ddot{x}(t) = \varepsilon P, \quad (15)$$

$$\ddot{y}(t) = \left(I_2 - \frac{\partial Y}{\partial \dot{y}} \right)^{-1} \left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} \dot{y} + \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial \varphi_1}{\partial t} ds \right) + \varepsilon Q,$$

где

$$\begin{aligned}
 P = P(t, x, y, \dot{x}, \dot{y}, \varepsilon) &= \left(I_1 - \varepsilon \frac{\partial X}{\partial \dot{x}} \right)^{-1} \left\{ I_1 + \varepsilon \frac{\partial X}{\partial \dot{y}} \left[I_2 - \frac{\partial Y}{\partial \dot{y}} - \right. \right. \\
 &- \varepsilon \frac{\partial Y}{\partial \dot{x}} \left(I_1 - \varepsilon \frac{\partial X}{\partial \dot{x}} \right)^{-1} \frac{\partial X}{\partial \dot{y}} \left. \right]^{-1} \frac{\partial Y}{\partial \dot{x}} \left(I_1 - \varepsilon \frac{\partial X}{\partial \dot{x}} \right)^{-1} \left\{ \left[\frac{\partial X}{\partial t} + \frac{\partial X}{\partial y} Y + \right. \right. \\
 &+ \frac{\partial X}{\partial z} \int_0^1 \frac{\partial \varphi}{\partial t} ds + \varepsilon \frac{\partial X}{\partial x} X \left. \right] + \left(I_1 - \varepsilon \frac{\partial X}{\partial \dot{x}} \right)^{-1} \frac{\partial X}{\partial \dot{y}} \left[I_2 - \frac{\partial Y}{\partial \dot{y}} - \right. \\
 &- \varepsilon \frac{\partial Y}{\partial \dot{x}} \left(I_1 - \varepsilon \frac{\partial X}{\partial \dot{x}} \right)^{-1} \frac{\partial X}{\partial \dot{y}} \left. \right]^{-1} \left[\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} Y + \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial \varphi_1}{\partial t} ds + \varepsilon \frac{\partial Y}{\partial x} X \right] \left. \right\}, \\
 Q = Q(t, x, y, \dot{x}, \dot{y}, \varepsilon) &= \left(I_2 - \frac{\partial Y}{\partial \dot{y}} \right)^{-1} \left\{ \frac{\partial Y}{\partial x} X + \frac{\partial Y}{\partial \dot{x}} \left(I_1 - \varepsilon \frac{\partial X}{\partial \dot{x}} \right)^{-1} \right. \\
 &\left. \left[\frac{\partial X}{\partial t} + \frac{\partial X}{\partial y} Y + \frac{\partial X}{\partial z} \int_0^1 \frac{\partial \varphi}{\partial t} ds + \varepsilon \frac{\partial X}{\partial x} X \right] \right\} + \frac{O(\varepsilon)}{\varepsilon} \left\{ \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} Y + \right. \\
 &+ \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial \varphi_1}{\partial t} ds + \varepsilon \frac{\partial Y}{\partial x} X + \varepsilon \frac{\partial Y}{\partial \dot{x}} \left(I_1 - \varepsilon \frac{\partial X}{\partial \dot{x}} \right)^{-1} \left. \right\} \\
 &\left. \left[\frac{\partial X}{\partial t} + \frac{\partial X}{\partial y} Y + \frac{\partial X}{\partial z} \int_0^1 \frac{\partial \varphi}{\partial t} ds + \varepsilon \frac{\partial X}{\partial x} X \right] \right\}.
 \end{aligned}$$

Подставляя (15) в (13) получаем

$$\begin{aligned}
 \dot{\eta}(t) - \dot{\xi}(t) &= -\varepsilon \left[\frac{\partial W}{\partial t} + \frac{\partial W}{\partial y} Y + \frac{\partial W}{\partial \dot{y}} \left(I_2 - \frac{\partial Y}{\partial \dot{y}} \right)^{-1} \left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} Y + \right. \right. \\
 &+ \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial \varphi_1}{\partial t} ds \left. \right) - X(t, x, y, \dot{x}, \dot{y}, z) + \bar{X}(x, \dot{x}) \left. \right] + \\
 &+ \varepsilon [\bar{X}(\eta, \dot{\eta}) - \bar{X}(\xi, \dot{\xi})] + \varepsilon [\bar{X}(x, \dot{x}) - \bar{X}(\eta, \dot{\eta})] - \varepsilon^2 \left(\frac{\partial W}{\partial x} X + \right. \\
 &+ \frac{\partial W}{\partial \dot{x}} P + \frac{\partial W}{\partial \dot{y}} Q \left. \right).
 \end{aligned} \tag{16}$$

В силу условий теоремы

$$\|\dot{\eta}(t) - \dot{\xi}(t)\| \leq \varepsilon c [\|\eta(t) - \xi(t)\| + \|\dot{\eta}(t) - \dot{\xi}(t)\| + \|\eta(t) - x(t)\| + \|\dot{\eta}(t) - \dot{x}(t)\|] + \varepsilon^2 M, \quad (17)$$

где

$$\left\| \frac{\partial W}{\partial x} X + \frac{\partial W}{\partial \dot{x}} P + \frac{\partial W}{\partial \dot{y}} Q \right\| \leq M = \text{const}... \quad (18)$$

Имея ввиду, что

$$\|\eta(t) - x(t)\| = \varepsilon \|\dot{W}\| \leq \varepsilon M_1, \quad (19)$$

$$\|\dot{\eta}(t) - \dot{x}(t)\| = \varepsilon \|\dot{W}\| \leq \varepsilon M_2, \quad (20)$$

где M_1 и M_2 — положительные постоянные, получаем

$$\|\dot{\eta}(t) - \dot{\xi}(t)\| \leq \frac{\varepsilon c}{1 - \varepsilon c} \|\eta(t) - \xi(t)\| + \frac{\varepsilon^2(cM_1 + cM_2 + M)}{1 - \varepsilon c}, \quad (21)$$

$$[\eta(t) - \xi(t)]|_{t=0} = 0$$

Отсюда следует, что

$$\begin{aligned} \|\eta(t) - \xi(t)\| &\leq \frac{\varepsilon^2(cM_1 + cM_2 + M)}{1 - \varepsilon c} \int_0^t \exp\{\varepsilon c(1 - \varepsilon c)^{-1}(t - \tau)\} d\tau = \\ &= \varepsilon c^{-1} [\exp\{\varepsilon c(1 - \varepsilon c)^{-1}t\} - 1](cM_1 + cM_2 + M). \end{aligned} \quad (22)$$

На отрезке $0 \leq t \leq T$ имеем оценку

$$\|\eta(t) - \xi(t)\| \leq \varepsilon c^{-1}(cM_1 + cM_2 + M) [\exp\{cL(1 - \varepsilon c)^{-1}\} - 1]. \quad (23)$$

Учитывая выражение для $\eta(t)$ находим

$$\begin{aligned} \|x(t) - \xi(t)\| &\leq \|x(t) - \eta(t)\| + \|\eta(t) - \xi(t)\| \leq \varepsilon \|W\| + \\ &+ \varepsilon c^{-1}(cM_1 + cM_2 + M) [\exp\{cL(1 - \varepsilon c)^{-1}\} - 1]. \end{aligned} \quad (24)$$

Так как функция $W(t)$ ограничена, то выражение, стоящее в правой части неравенства (24) при достаточно малом ε ($0 < \varepsilon \leq \varepsilon^0$, $\varepsilon^0 \leq \varepsilon^*$) может быть сделано сколь угодно малым, т.е., справедливо неравенство

$$\|x(t) - \xi(t)\| < \delta.$$

Этим теорема 1 доказана.

Перейдем теперь к обоснованию второго варианта метода усреднения.

Пусть вдоль интегральных кривых $y = \psi(t, x, \dot{x}, \lambda, T)$ краевой задачи (3), (4), где λ рассматривается как векторный параметр, существуют независимые от λ средние значения

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t, x, \psi(t, x, \dot{x}, \lambda, T), \dot{x}, \frac{\partial}{\partial t} \psi(t, x, \dot{x}, \lambda, T), z) dt = \bar{X}(x, \dot{x}, z), \quad (25)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(t, s, x, \psi(s, x, \dot{x}, \lambda, T), \dot{x}, \frac{\partial}{\partial s} \psi(s, x, \dot{x}, \lambda, T)) ds = \bar{\varphi}(t, x, \dot{x}). \quad (26)$$

Тогда усредненным уравнением первого приближения для медленных переменных $x(t)$ решения $\{x(t), y(t)\}$ краевой задачи (1), (2) назовем уравнение

$$\dot{\xi}(t) = \varepsilon \bar{X}(\xi(t), \xi(t), \int_0^1 \bar{\varphi}(t, \xi(s), \xi(s)) ds) \quad (27)$$

с начальным условием

$$\xi(0) = x^0. \quad (28)$$

Докажем теорему о близости компонент $x(t)$ решения $\{x(t), y(t)\}$ краевой задачи (1), (1) и решения $\xi(t)$ задачи Коши (27), (28).

ТЕОРЕМА 2. Пусть:

1. Выполнены условия 1, 2, 3 и 5 теоремы 1,
2. Функция $\varphi(t, s, x, y, u, v)$ удовлетворяет в области $G^*(t, s, x, y, u, v)$ неравенству

$$\|\varphi(t, s, x, y, u, v) - \varphi(t, s, x', u', v)\| \leq \sigma_1(t, s) [\|x - x'\| + \|u - u'\|],$$

где функция $\sigma_1(t, s)$ со своей стороны удовлетворяет условию

$$\int_0^1 \sigma_1(t, s) ds \leq c_1 = \text{const.}$$

3. Для всех $(x, u, z, \lambda) \in G^* (x, u, z) \times \Lambda$ и $(t, x, u, \lambda) \in G^*(t, x, u) \times \Lambda$ существуют независимые от параметра λ пределы

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t, x, \psi(t, x, u, \lambda, T), \lambda, \frac{\partial}{\partial t} \psi(t, x, u, \lambda, T), z) dt = \bar{X}(x, u, z), \quad (29)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(t, s, x, \psi(t, x, u, \lambda, T), u, \frac{\partial}{\partial s} \psi(s, x, u, \lambda, T)) ds = \bar{\varphi}(t, x, u), \quad (30)$$

причем предельные переходы в (29) и (30) происходят равномерно относительно совокупности $(x, u, z, \lambda) \in G^*(x, u, z) \times \Lambda$ и $(t, x, \lambda, u) \in G^*(t, x, u) \times \Lambda$. Средние значения $\bar{X}(x, u, z)$ и $\bar{\varphi}(t, x, u)$ определены, непрерывны и равномерно ограничены для всех $(x, u, z) \in G^* (x, u, z)$ и $(t, x, u) \in G^* (t, x, u)$ и удовлетворяют условиям

$$\|\bar{X}(x, u, z) - \bar{X}(x', u', z')\| \leq c [\|x - x'\| + \|u - u'\| + \|z - z'\|],$$

$$\|\bar{\varphi}(t, x, u)\| \leq \sigma_2(t),$$

$$\|\varphi(t, s, x, y, u, v) - \bar{\varphi}(t, x, u)\| \leq \sigma_3(t, s),$$

где c — положительная постоянная, а функции $\sigma_2(t)$ $\sigma_3(t, s)$ удовлетворяют условиям

$$\frac{1}{t} \int_0^t \sigma_2(\tau) d\tau \rightarrow 0 \quad \text{при } t \rightarrow \infty,$$

$$\frac{1}{t} \int_0^t d\tau \int_0^1 \sigma_3(\tau, s) ds \rightarrow 0 \quad \text{при } t \rightarrow \infty.$$

4. Решение усредненного уравнения

$$\dot{\xi}(t) = \varepsilon \bar{X}(\xi(t), \xi(t), \int_0^1 \bar{\varphi}(t, \xi(s), \xi(s)) ds) \quad (31)$$

удовлетворяющее начальному условию $\xi(0) = x^0$ существует и однозначно определено при $t \geq 0$ для всех значений $\varepsilon \in (0, \varepsilon^*]$ и кривая $\{\xi(t), \xi(t)\}$ соответствующая этому решению, на этом интервале не выходит из области $G^*(x, u)$.

5. Уравнение в частных производных

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial y} Y + \frac{\partial W}{\partial v} \left(I_2 - \frac{\partial Y}{\partial v} \right)^{-1} \left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} Y + \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial \varphi_1}{\partial t} ds \right) =$$

$$= X(t, x, y, u, v, z) - \bar{X}(x, u, z), \quad (32)$$

$$W|_{t=0} = 0, \quad x = \text{const.}, \quad u = \text{const.},$$

обладает некоторым решением $W = W(t, x, y, u, v)$ которое определено и равномерно ограничено вместе со своими частными производными первого порядка во всей области G^* .

Тогда если $\{x(t), y(t)\}$ решение краевой задачи (1), (2), а $\xi(t)$ — решение усредненного уравнения (31) с начальным условием $\xi(0) = x^0$, то для любого сколь угодно малого δ можно указать такое $\varepsilon^0 > 0$ что при $0 < \varepsilon \leq \varepsilon^0$ на отрезке $0 \leq t \leq T$ будет выполняться неравенство

$$\|x(t) - \xi(t)\| < \delta.$$

Доказательство. Оценим разность $\eta(t) - \xi(t)$ где $\xi(t)$ — решение усредненного уравнения (31) с начальным условием $\xi(0) = x^0$, а $\eta(t)$ — выражение (12).

Имеем

$$\eta(t) - \xi(t) = \dot{x}(t) - \varepsilon \left\{ \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} \dot{x}(t) + \frac{\partial W}{\partial y} \dot{y}(t) + \frac{\partial W}{\partial \dot{x}} \ddot{x}(t) + \right.$$

$$\left. + \frac{\partial W}{\partial \dot{y}} \ddot{y}(t) + \bar{X}(\xi(t), \dot{\xi}(t), \zeta(t)) \right\}, \quad (33)$$

где

$$\zeta(t) = \int_0^1 \dot{\varphi}(t, \xi(s), \dot{\xi}(s)) ds.$$

Продифференцировав систему (1) и подставив (15) в (33) получаем

$$\eta(t) - \xi(t) = -\varepsilon \left[\frac{\partial W}{\partial t} + \frac{\partial W}{\partial y} Y + \frac{\partial W}{\partial \dot{y}} \left(I_2 - \frac{\partial Y}{\partial \dot{y}} \right)^{-1} \left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} Y + \right. \right.$$

$$\left. + \frac{\partial Y}{\partial w} \int_0^1 \frac{\partial \varphi_1}{\partial t} ds \right) - X(t, x, y, \dot{x}, \dot{y}, z) + \bar{X}(x, \dot{x}, z) \Big] +$$

$$+ \varepsilon [\bar{X}(\eta, \dot{\eta}, \zeta) - \bar{X}(\xi, \dot{\xi}, \zeta)] + \varepsilon [\bar{X}(x, \dot{x}, z) - \bar{X}(\eta, \dot{\eta}, \zeta)] +$$

$$- \varepsilon^2 \left(\frac{\partial W}{\partial x} X + \frac{\partial W}{\partial \dot{x}} P + \frac{\partial W}{\partial \dot{y}} Q \right).$$

В силу условий теоремы 2 и неравенств (18), (19) и (20) находи

$$\|\dot{\eta}(t) - \dot{\xi}(t)\| \leq \frac{\varepsilon c}{1 - \varepsilon c} \|\eta(t) - \xi(t)\| + \frac{\varepsilon^2}{1 - \varepsilon c} [c(M_1 + M_2)(1 + c_1) + M] +$$

$$+ \frac{\varepsilon c}{1 - \varepsilon c} [2\sigma_2(t) + \sigma_{30}(t)],$$

где

$$\sigma_{30}(t) = \int_0^1 \sigma_3(t, s) ds.$$

Отсюда, так как $\eta(0) = \xi(0)$ следует

$$\|\eta(t) - \xi(t)\| \leq \frac{1}{1 - \varepsilon c} \int_0^t \{ \varepsilon^2 [c(M_1 + M_2)(1 + c_1) + M] +$$

$$+ \varepsilon c [2\sigma_2(\tau) + \sigma_{30}(\tau)] \} \exp \{ \varepsilon c (1 - \varepsilon c)^{-1} (t - \tau) \} d\tau.$$

Введя функции

$$\gamma_1(t) = \frac{1}{t} \int_0^t \sigma_2(\tau) d\tau \rightarrow 0 \quad \text{при } t \rightarrow \infty,$$

$$\gamma_2(t) = \frac{1}{t} \int_0^t \sigma_{30}(\tau) d\tau \rightarrow 0 \quad \text{при } t \rightarrow \infty,$$

находим, что на отрезке $0 \leq t \leq T$

$$\|\eta(t) - \xi(t)\| \leq \frac{1}{1 - \varepsilon c} \exp \left\{ \frac{cL}{1 - \varepsilon c} \right\} \{ \varepsilon [c(M_1 + M_2)(1 + c_1) +$$

$$+ M] L + cL [2\gamma_1(L\varepsilon^{-1}) + \gamma_2(L\varepsilon^{-1})] \}.$$

Из этого неравенства и из выражения (12) для $\eta(t)$ видно, что всегда можно выбрать ε ($0 < \varepsilon \leq \varepsilon^0$, $\varepsilon^0 \leq \varepsilon^*$) таким образом чтобы на отрезке $0 \leq t \leq T$ было справедливо неравенство

$$\|x(t) - \xi(t)\| < \delta.$$

Этим теорема 2 доказана.

Пловдивский университет
Имени П. Хилендарского

Высший машино электротехнический
Институт имени В. И. Ленина София

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Soit $u : A \rightarrow B$ un morphisme d'anneaux commutatifs unitaires. On dit qu'un idéal premier $P \subset B$ est lisse sur A si le morphisme $u' : A \rightarrow B$ est lisse, où $Q = u^{-1}(P)$. On note d'habitude avec $\text{Lis}(A, B)$ l'ensemble des idéaux premiers de B lisses sur A . Un des problèmes de l'algèbre locale est de donner des conditions pour que l'ensemble $\text{Lis}(A, B)$ soit ouvert dans $\text{Spec}(B)$ ou bien pour que l'ensemble $\text{Lis}(A, B)$ contienne un sous-ensemble ouvert non vide de $\text{Spec}(B)$ (voir [2] et [3]). La plupart des résultats de ce genre utilisent la théorie différentielle de la lissité, surtout les critères jacobiens de Zariski et Nagata. Le but de ce note est de démontrer un résultat de cette sorte en utilisant la technique des p -bases. Au commencement on va démontrer une condition suffisante pour que $\text{Lis}(A, B)$ contienne un ouvert non vide de $\text{Spec}(B)$. Ensuite on va déduire de ce résultat un théorème de Kunz en obtenant ainsi une nouvelle démonstration assez simple de ce théorème. Enfin on va donner une application du résultat en ce qui concerne la lissité des sous-algèbres d'une algèbre de type fini.

1. Un critère de lissité

Tous les anneaux sont supposés commutatifs et unitaires. Les notations et la terminologie sont celles de [1], [2] et [3]. On va donner encore quelques définitions :

Définition 1.

Soit $u : A \rightarrow B$ un morphisme d'anneaux de caractéristique $p \neq 0$, p premier. On va dire que u est séparable si le morphisme canonique

$$\psi : A^{(p)} \otimes_A B \rightarrow B^{(p)}$$

est injectif.

Notons que si A est un corps, la Définition 1 coïncide avec la définition classique d'une algèbre séparable sur un corps. En effet cela résulte de la suivante.

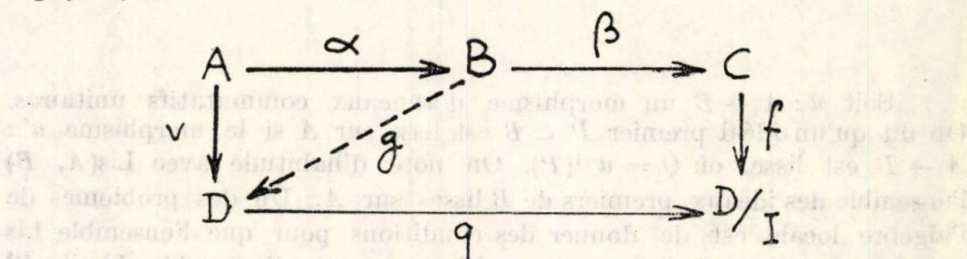
Observation 1.

Si l'anneau $A^{(p)} \otimes_A B$ est réduit alors B est une A -algèbre séparable.

Démonstration. Soit $x = \sum \lambda_i \otimes b_i \in A^{(p)} \otimes_A B$ tel que $\psi(x) = 0$. On a alors $0 = \psi(x) = \sum \lambda_i b_i^p$, par la définition de ψ . Il en résulte $0 = 1 \otimes \sum \lambda_i b_i^p = \sum 1 \otimes \lambda_i b_i^p = \sum \lambda_i^p \otimes b_i^p = x^p$. Puisque $A^{(p)} \otimes_A B$ est réduit, on déduit que $x = 0$.

Définition 2. Soit $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ morphismes d'anneaux. On va dire que le morphisme α est lisse dominé par β (ou bien que B est une A -algèbre lisse dominé par V) si quelque soit les morphismes d'anneaux $v : A \rightarrow D$,

$q: D \rightarrow D/I$ $f: C \rightarrow D/I$, avec $f \circ \beta \circ \alpha = q \circ v$, et I étant un idéal de D , $I^2 = 0$, il existe un morphisme d'anneaux $g: B \rightarrow D$ tel que $g \circ \alpha = v$ et $q \circ g = f \circ \beta$



On va démontrer maintenant une proposition analogue à [2], vol. II, chapitre IX, 2.3, mais plus précise :

Proposition 1. Soit $u: A \rightarrow B$ un morphisme séparable d'anneaux de caractéristique $p \neq 0$, p premier et soit $C = A[B^p][F] \subset B$ l'algèbre générée sur A par B^p et par une partie F de B , p -libre sur A (eventuellement vide). Alors C est une A -algèbre lisse dominée par B .

Démonstration.

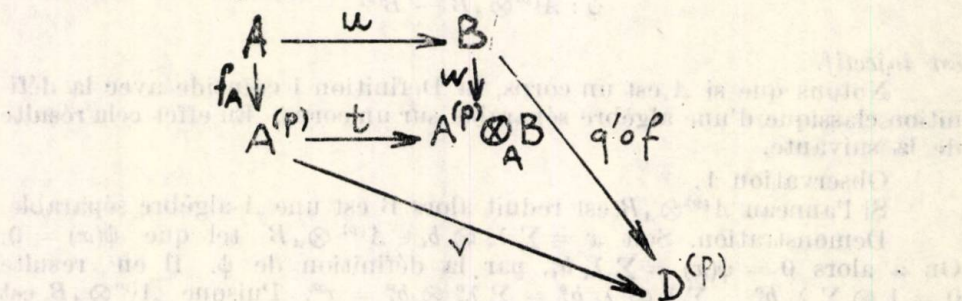
Soit $v: A \rightarrow D$, $q: D \rightarrow E$, $f: B \rightarrow E$ morphismes d'anneaux, ou $E = D/I$ et $I^2 = 0$, tel que $f \circ u = q \circ v$. Pour tout anneau R de caractéristique $p \neq 0$, p premier, soit $f_R: R \rightarrow R^{(p)}$ le morphisme de Frobenius, $f_R(x) = x^p$. Puisque $(\text{Ker}(q))^2 = 0$, il en résulte que $\text{Ker}(q) \subset \text{Ker}(f_D)$ et donc, il existe $q': E \rightarrow D^{(p)}$ un morphisme d'anneaux tel que

$$(I) \quad q' \circ q = f_D.$$

Le morphisme $q: D \rightarrow E$ se prolonge canoniquement en un morphisme $D^{(p)} \rightarrow E^{(p)}$ que nous noterons toujours avec q . Puisque le morphisme composé $D \xrightarrow{q} E \xrightarrow{q'} D^{(p)} \xrightarrow{q} E^{(p)}$ est égal de (I) avec le morphisme composé $D \xrightarrow{q} E \xrightarrow{f_E} E^{(p)}$ et puisque q est surjectif, on déduit que

$$(II) \quad q \circ q' = f_E$$

Considérons le diagramme



où t et w sont les morphismes canoniques. Ce diagramme est commutatif car $q' \circ f \circ u = q' \circ q \circ v = f_D \circ v = v \circ f_A$. Alors il existe un morphisme d'anneaux $g: A^{(p)} \otimes_A B \rightarrow D^{(p)}$ tel que $g \circ w = q' \circ f$ et $g \circ t = v$. Il en résulte que le diagramme

$$\begin{array}{ccccc}
 u : A^{(p)} & \xrightarrow{t} & A^{(p)} \otimes_A B & \xrightarrow{\psi} & B^{(p)} \\
 & \searrow v & \downarrow g & & \downarrow f \\
 & & D^{(p)} & \xrightarrow{q} & E^{(p)}
 \end{array}$$

est commutatif. En effet, $q \circ g(\lambda \otimes a) = q(v(\lambda)q'(f(a))) = (q \circ v(\lambda))(q \circ q'(f(a))) = f \circ u(\lambda)(f(a))^p = f(u(\lambda)a^p) = f \circ \psi(\lambda \otimes a)$. Si $F = \emptyset$, la proposition est démontrée. Si $F = (x_i)_{i \in I}$ il en résulte de la construction de C que C est un $A[B^p]$ module libre ayant comme base les produits finis $\prod x_i^{n_i}$, $0 \leq n_i < p$. Puisque ψ est injectif par l'hypothèse on déduit que C est un $A^{(p)} \otimes_A B$ module libre avec la structure induite par ψ . Soit $\psi_1 : A^{(p)} \otimes_A B \rightarrow C$ la corestriction de ψ à C et soit $\psi_2 : C \rightarrow B$ le morphisme d'inclusion. Pour chaque $i \in I$, soit $\alpha_i \in q^{-1}(f \circ \psi_2(x_i))$. Il existe alors un morphisme s de $A^{(p)} \otimes_A B$ -modules, $s : C \rightarrow D$, tel que $s(\prod x_i^{n_i}) = \prod \alpha_i^{n_i}$. Alors $s \circ \psi_1 = g$ et $q \circ s = f \circ \psi_2$. Mais s est aussi un morphisme d'anneaux car $s(\prod x_i^{n_i} \prod x_i^{m_i}) = s(\prod x_i^{n_i+m_i}) = s((\psi_1(\prod 1 \otimes x_i)) \prod x_i^{k_i}) = g(\prod 1 \otimes x_i) \prod \alpha_i^{k_i} = g(w(\prod x_i)) \prod \alpha_i^{k_i} = q' \circ f(\prod x_i) \prod \alpha_i^{k_i}$ ou $k_i \equiv m_i + n_i \pmod{p}$, $k_i < p$. Mais $\alpha_i \in q^{-1}(f(x_i))$ implique $q(\alpha_i) = f(x_i)$ donc $q' \circ q(\alpha_i) = q' \circ f(x_i)$. On déduit que $\alpha_i^p = q' \circ f(x_i)$. Alors $q' \circ f(\prod x_i) \prod \alpha_i^{k_i} = \prod \alpha_i^p \prod \alpha_i^{k_i} = s(\prod x_i^{n_i}) s(\prod x_i^{m_i})$ ce qui achève la démonstration. Notons qu'il ne résulte pas de ces considérations que C soit une algèbre p -libre sur A .

On verra maintenant combien de large peut être choisie la partie F de B .

Proposition 2.

Soit $u : A \rightarrow B$ un morphisme séparable d'anneaux de caractéristique $p \neq 0$, p premier et B intègre. Alors u se factorise

$$\begin{array}{ccc}
 A & \xrightarrow{u} & B \\
 & \searrow v & \nearrow i \\
 & C &
 \end{array}$$

où i est un morphisme d'inclusion et on a

- Les anneaux B et C ont le même corps des fractions
- Le morphisme v est lisse dominé par i .

Démonstration.

Soit Σ l'ensemble des parties de B p -libre sur A . Supposons pour commencement que $\Sigma \neq \emptyset$. On voit aisément que Σ est inductivement ordonnée par rapport à l'inclusion. Soit F un élément maximal de Σ et posons $C = A[B^p][F]$. De la Proposition 1, il en résulte (b). Pour démontrer (a), il suffit que nous montrons $B \subset Q(C)$, où $Q(C)$ est le corps des fractions de C . En effet, s'il existait un élément $h \in B \setminus Q(C)$, alors la famille $F \cup \{h\}$ serait p -libre sur A (si non, il existerait une relation de dépendance linéaire entre $1, h, h^2, \dots, h^{p-1}$ sur $A[B^p][F] = C$ donc sur



$Q(C)$. Puisque $h \notin Q(C)$, il en résulte que le polynôme $X^p - h^p$ est irréductible sur $Q(C)$, absurde). L'hypothèse de l'existence de h nous conduit, donc, au fait que F n'est pas maximal en Σ , et ainsi l'inclusion $B \subset Q(C)$ est démontrée.

Si $\Sigma = \emptyset$ on obtient que pour tout $h \in B$, il existe une relation de dépendance linéaire entre $1, h, h^2, \dots, h^{p-1}$ sur $A[B^p]$. En notant $C = A[B^p]$ on déduit d'une manière analogue $Q(B) = Q(C)$. D'autre part $A[B^p]$ est une A -algèbre lisse dominée par B , ce qui résulte de la Proposition 1 et ainsi la Proposition 2 est démontrée.

Nous sommes maintenant préparés pour démontrer le résultat principal de cette note.

Proposition 3

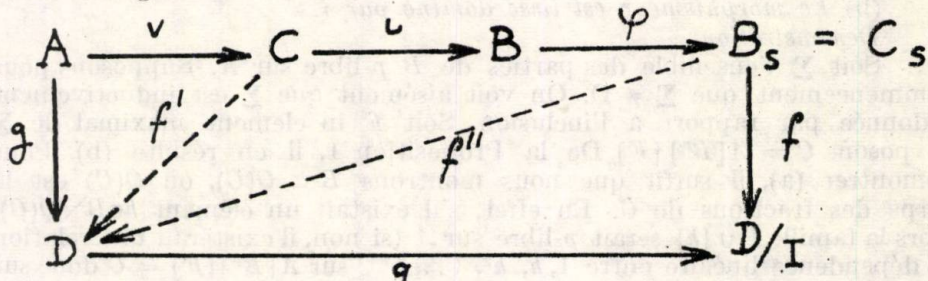
Soit $u: A \rightarrow B$ un morphisme séparable d'anneaux de caractéristique $p \neq 0$, p premier. Supposons que B soit intègre et qu'il existe une $A[B^p]$ -algèbre de type fini intercalée entre B et le corps des fractions de B . Alors, il existe un élément $s \in B$, $s \neq 0$ tel que le morphisme

$$A \rightarrow B_s \text{ soit lisse}$$

Démonstration. Soit E une $A[B^p]$ -algèbre de type fini telle que $B \subset E \subset Q(B)$. Il résulte de la Proposition 2 que le morphisme u se factorise $u: A \xrightarrow{v} C \xrightarrow{i} B$ où i est un morphisme d'inclusion, C est une A -algèbre lisse dominée par B et $Q(B) = Q(C)$. Soit $\frac{c_1}{s_1}, \frac{c_2}{s_2}, \dots, \frac{c_n}{s_n}$ engendrant E comme $A[B^p]$ -algèbre. Puisqu'ils appartiennent à $Q(B) = Q(C)$, on peut supposer que les éléments $c_1, c_2, \dots, c_n, s_1, s_2, \dots, s_n$ appartiennent à C . Posons $s = s_1 s_2 \dots s_n$ et montrons que $B_s = C_s$. En effet, puisque $C \subset B$ on déduit $C_s \subset B_s$. Pour montrer que $B_s \subset C_s$ il suffit de démontrer que $B \subset C_s$. Soit $x \in B$. Alors il existe un polynôme R en n variables à coefficients dans C tel que

$$x = R\left(\frac{c_1}{s_1}, \frac{c_2}{s_2}, \dots, \frac{c_n}{s_n}\right)$$

Alors si k est le degré du polynôme R on a $s^k x \in C$, c'est-à-dire $x \in C_s$. Soit $g: A \rightarrow D, q: D \rightarrow D/I, f: B_s \rightarrow D/I$, morphismes d'anneaux $I \subset D$ étant un idéal, $I^2 = 0$, tel que le morphisme composé $A \xrightarrow{u} B \xrightarrow{\varphi} B_s \xrightarrow{f} D/I$ soit égal à $A \xrightarrow{g} D \xrightarrow{q} D/I$ où φ est le morphisme canonique. Alors on a le suivant diagramme commutatif :



Du fait que v est lisse dominé par i , il en résulte qu'il existe un morphisme d'anneaux $f' : C \rightarrow D$ tel que $f' \circ v = g$ et $q \circ f' = f \circ \varphi \circ i$. Du fait que le morphisme $\varphi \circ i : C \rightarrow C_s$ est le morphisme structurel d'une algèbre des fractions et donc $\varphi \circ i$ est lisse, il en résulte qu'il existe un morphisme d'anneaux $f'' : B_s = C_s \rightarrow D$ tel que $f'' \circ \varphi \circ i = f'$ et $q \circ f'' = f$. Enfin il résulte que $f'' \circ \varphi \circ i \circ v = f' \circ v = g$, ce qui achève la démonstration.

Corollaire 1.

Soit $u : A \rightarrow B$ un morphisme séparable d'anneaux de caractéristique $p \neq 0$, p premier, tel que B soit intègre. Supposons qu'une des conditions suivantes soit satisfaite :

(i) $[B : B^p] < \infty$

(ii) u est de type fini

Alors il existe un élément $s \in B$, $s \neq 0$ tel que le morphisme $A \rightarrow B$, soit lisse.

Démonstration. Il suffit d'observer que dans les deux cas, B est une $A[B^p]$ — algèbre de type fini est évidemment B est intercalé entre B et $Q(B)$. Alors le corollaire résulte directement de la Proposition 3.

Corollaire 2. Dans les hypothèses de la Proposition 3, ou bien dans celles du Corollaire 1, il en résulte que $\text{Lis}(A, B)$ contient un sous-ensemble ouvert non vide de $\text{Spec}(B)$.

2. Sur un théorème de Kunz

Soit A un anneau noethérien. On dit qu'un idéal premier $P \subset A$ est régulier si l'anneau local A_P est régulier. On note avec $\text{Reg}(A)$ l'ensemble des idéaux réguliers de A . On va déduire de la proposition 3 le suivant

Théorème (Kunz) Soit A un anneau noethérien de caractéristique $p \neq 0$, p premier. Si $[A : A^p] < \infty$, alors $\text{Reg}(A)$ est ouvert dans $\text{Spec}(A)$. On a besoin de deux résultats classiques :

Lemme 1.

Un anneau local noethérien qui contient un corps est régulier si et seulement s'il est une algèbre formellement lisse sur son corps premier.

Pour démonstration, voir [1], IV, ou bien [2], II, chapitre IX.

Lemme 2. Soit A un anneau noethérien et supposons que pour tout idéal premier régulier $P \subset A$, $\text{Reg}(A/P)$ contient un sous-ensemble non vide et ouvert dans $\text{Spec}(A/P)$. Alors $\text{Reg}(A)$ est ouvert dans $\text{Spec}(A)$.

Pour démonstration, voir [1], IV, ou bien [2], II, chapitre XII.

Démonstration du théorème.

Le lemme 2 montre qu'il suffit de démontrer que pour tout idéal premier $P \subset A$, $\text{Reg}(A/P)$ contient un sous-ensemble non vide de $\text{Spec}(A/P)$. Il suffit du Lemme 1 de démontrer que pour tout idéal premier $P \subset A$, $\text{Lis}(F_p, A/P)$ contient un sous-ensemble non vide ouvert de $\text{Spec}(A/P)$. Or, cette affirmation résulte du Corollaire 2 de la Proposition 3, tenant compte que le morphisme $F_p \rightarrow A/P$ est séparable et que A/P est intègre et fini sur $(A/P)^p$.

Pour une autre démonstration, voir [3].

3. Sur la lissité des sous-algèbres d'une algèbre de type fini

Soit $u: A \rightarrow B$ un morphisme de type fini d'anneaux, B intègre, et considérons les sous- A -algèbres de B qui ont le même corps des fractions que celui de B . On va démontrer que ces sous-algèbres jouissent d'une propriété remarquable de lissité. La théorie classique nous donne peu d'informations sur tels anneaux qui sont en général non noethériens et qui ont une structure difficile à contrôler.

Proposition 4.

Soit A un anneau de caractéristique $p \neq 0$, p premier, et soit B une A -algèbre intègre de type fini. Soit C une sous- A algèbre séparable de B dont le corps des fractions est égal à celui de B . Alors il existe un élément $s \in C$, $s \neq 0$, tel que C_s soit une A -algèbre lisse.

Démonstration.

La proposition découle de la Proposition 3 car B est une $A[C^p]$ -algèbre de type fini, intercalée entre C et le corps des fractions de C , qui étant égal à $Q(B)$, contient B .

Corrolaire 1

Soit k un corps de caractéristique $p \neq 0$ et soit B une k -algèbre intègre de type fini dont le corps des fractions K est une extension séparable de k . Alors pour tout sous- k -algèbre C de B qui a le corps des fractions égal à K , il existe un élément $s \in C$, $s \neq 0$, tel que C_s soit une k -algèbre lisse.

Démonstration

L'extension $k \subset K$ étant séparable par l'hypothèse, il en résulte que l'anneau $k^{(p)} \otimes_k K$ est réduit. Du fait que $k^{(p)}$ est une k -algèbre plate, on déduit que $k^{(p)} \otimes_k C$ est un sous-anneau de $k^{(p)} \otimes_k K$, donc il est réduit. Il en résulte comme on l'a déjà observé que C doit être une k -algèbre séparable et on peut appliquer la Proposition 4.

Corrolaire 2

Dans les hypothèses du Corrolaire 1, il en résulte que $\text{Lis}(k, C)$ contient un sous-ensemble non vide ouvert de $\text{Spec}(C)$.

Corrolaire. 3. Soit k un corps de caractéristique $p \neq 0$. Pour tout sous- k -algèbre de $k[T_1, T_2, \dots, T_n]$ dont le corps des fractions est $k(T_1, T_2, \dots, T_n)$, $\text{Lis}(k, C)$ contient un sous-ensemble non vide ouvert de $\text{Spec}(C)$.

Finissons par donnant un exemple non trivial pour le dernier corollaire. Soit k un corps de caractéristique $p \neq 0$ et soit $B = k[X, Y]$. Soit C la k -algèbre engendrée sur k par les éléments de la forme $X^n Y^m$, où $n = 0, 1, 2, \dots$ et $0 \leq m \leq n^2$. L'anneau C n'est pas noethérien et il satisfait les hypothèses du Corrolaire 3.

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Faculty of Mathematics
University of Bucharest

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Introduction

The aim of the present research is to present an algorithm to find the maximum flow in a network, corresponding to several problems raised in practice, among which we mention : further modifications of the capacity of arcs, the change of source ou output in other nodes of the network and others.

At the same time the algorithm proposed is more general and seems adequate for extension to the flow of several products.

Basic notions

Before describing the mentioned algorithm, we shall present the definitions, notations and the necessary results.

By *transport network*, with $n + 2$, vertexes (nodes), is meant the oriented connex graph without $G_{n+2} = (X, \mathcal{A})$ with vertex $x_0 \in X$ called *source* and a vertex $x_{n+1} \in X$ called *destination* (output), together with a real strictly positive function, called *capacity function*, which attaches to each arc (x_i, x_j) the value c_{ij} .

We shall denote the transport network with $n + 2$ nodes by $G_{n+2} = (X, \mathcal{A}, x_0, x_{n+1}, c)$ or just by $G_{n+2}(c)$.

Is called *flow* on the network $G_{n+2}(c)$ a real nonnegative function φ , defined on the set of arcs, so that to any arc $(x_i, x_j) \in \mathcal{A}$ corresponds the value φ_{ij} and for which are accomplished the conditions :

- 1) $\sum_{(i,j) \in \mathcal{A}} \varphi_{ij} - \sum_{(j,i) \in \mathcal{A}} \varphi_{ji} = 0$, for $i = 1, 2, \dots, n$;
- 2) $0 \leq \varphi_{ij} \leq c_{ij}$, for $(i, j) \in \mathcal{A}$
- 3) $\sum_{(0,j) \in \mathcal{A}} \varphi_{0j} \geq \sum_{(j,0) \in \mathcal{A}} \varphi_{j0}$.

Is called *total value* of the flow which goes through the network from the source to the output, the value :

$$4) \sum_{(0,j) \in \mathcal{A}} \varphi_{0j} - \sum_{(j,0) \in \mathcal{A}} \varphi_{j0} = \Phi$$

In addition from (1) we can also write :

$$4') \sum_{(j,n+1) \in \mathcal{A}} \varphi_{j,n+1} - \sum_{(n+1,j) \in \mathcal{A}} \varphi_{n+1,j} = \Phi.$$

We call *elementary flow* on $G_{n+2}(c)$, the flow φ , for which any circuit of the network contains at least an arc with a null flow.

Theorem 1.

If on a network of flow one can determine a flow of total value Φ , then one can determine also an elementary flow of total value Φ .

Demonstration

If flow φ is not elementary, there exist a circuit of the network, on which all arcs have a non null flow. Lessening the flow from all these arcs, with a value equal to the smallest of the arcs' flows, which form the circuit, one obtains a function φ_1 , verifying also conditions (1)–(3), thus another flow on the network. In addition, it is easily noticed that the total value of the new flow is equal also with Φ . Applying successively operations of this kind on the given flow, one obtains an elementary flow of total value Φ . Indeed, the sum of the flows' values from all the arcs of the network, which initially is finite size, diminishes each time with a strict positive value, a process which cannot continue indefinitely.

Given $(P, \bar{P})$ a partition of the flows' set of nodes, such as $x_0 \in P$ and $x_{n+1} \in \bar{P}$. Is called *cut (section)* through the network, corresponding to this partition, the subset of arcs:

$$S(P, \bar{P}) = \{(x_i, x_j) / (x_i, x_j) \in \mathcal{A}, x_i \in P, x_j \in \bar{P}\}.$$

The capacity of a cut S is defined as follows:

$$C(S) = \sum_{(x_i, x_j) \in S} c_{ij}$$

It is shown that for any flow φ , of total value Φ and for any cut S , we have

$$\Phi = \sum_{\substack{x_i \in P \\ x_j \in P}} \varphi_{ij} - \sum_{\substack{x_j \in \bar{P} \\ x_i \in P}} \varphi_{ji}.$$

Considering the set of all flows φ , which can be defined in a network, flows of total value Φ , as well as the set of cuts S , in the network, of capacities $C(S)$, it is known that: $\max\{\Phi\} = \min\{C(S)\}^*$.

The extended network, R' , corresponding to a network, obtained from the given network, by adding the arc (x_{n+1}, x_0) , of infinite capacity (very big). Considering a flow φ on the network R , of total value Φ , we agree to associate to it a definite flow in the corresponding extended network, namely (x_{n+1}, x_0) a flow of value Φ . In this way, conditions (1) extend also on the nodes x_0 and x_{n+1} . In addition, the value of the total maximum flow in the extended network, will be considered like in the given network, taking no notice of the flow on the arc (x_{n+1}, x_0) . We agree at last, that the use of the arc (x_{n+1}, x_0) should not be taken into consideration in the definition of the elementary flow.

* Fulkerson's theorem

Lemma 1

If in a network R there exists a flow of total value

$$\Phi = \min \left\{ \sum_{(x_0, x_j) \in \mathcal{A}} c_{0j}; \sum_{(x_i, x_{n+1}) \in \mathcal{A}} c_{i, n+1} \right\},$$

then is it a flow of total maximum value in the network.

Lemma 2

The total maximum value of the flow in a network R' , does not change, if we consider as a capacity of the arc (x_{n+1}, x_0) the value of capacity of any cut from the network.

The conclusions of the two lemmas from above result from the observation that the value of any total flow cannot exceed the capacity of any cut in the network, as well as from considering two particular cuts in the network.

Following, we agree to consider like value of the capacity of arc (x_{n+1}, x_0) in the extended flow, the value:

$$c_{n+1,0} = \min \left\{ \sum_{(x_0, x_j) \in \mathcal{A}} c_{0j}; \sum_{(x_i, x_{n+1}) \in \mathcal{A}} c_{i, n+1} \right\}.$$

We agree to call *semi-flow* on the network R (respectively on the extended network R'), a function ψ , defined on the set of the arcs' network, which accomplishes the conditions (2). Denoting

$$\psi_{ij} = \psi_{(x_i, x_j)},$$

we shall have thus, for a semi-flow ψ :

$$0 \leq \psi_{ij} \leq c_{ij}, \text{ for } (x_i, x_j) \in \mathcal{A} \quad (2')$$

In the case of an extended flow, we shall call a *normal semi-flow* any semi-flow ψ , for which we have:

$$\psi_{n+1,0} = c_{n+1,0}.$$

With the aid of a fixed semi-flow ψ , we build-up another function, defined on the network's set of nodes and which we shall call a *surpluss function* relatively to the semi-flow ψ , attaching thus to every node x_i , the real value:

$$s_i = s(x_i) = \sum_{(x_i, x_j) \in \mathcal{A}} \psi_{ij} - \sum_{(x_j, x_i) \in \mathcal{A}} \psi_{ji} \quad i = 0, 1, 2, 3, \dots, n+1. \quad (5)$$

The surpluss attached to a node of the network can be positive, negative, or null.

It is easy to observe that, if for a semi-flow, from an extended network we have $s_i = 0$, for $i = 1, 2, 3, \dots, n$, this semi-flow represents a flow in the given network. Also, an identically null semi-flow, defined on an extended flow, is a flow defined on this network.

Theorem 2

The surpluss function, relative to a any semi-flow, defined on a network (extended or not), verifies the following relation :

$$\sum_{x_i \in X} s_i = 0$$

Demonstration. We have successively :

$$\begin{aligned} \sum_{x_i \in X} s_i &= \sum_{x_i \in X} \left(\sum_{(x_i, x_j) \in \mathcal{A}} \psi_{ij} - \sum_{(x_j, x_i) \in \mathcal{A}} \psi_{ji} \right) = \sum_{x_i \in X} \sum_{(x_i, x_j) \in \mathcal{A}} \psi_{ij} - \sum_{x_i \in X} \sum_{(x_j, x_i) \in \mathcal{A}} \psi_{ji} = \\ &= \sum_{(x_i, x_j) \in \mathcal{A}} \sum_{x_i \in X} \psi_{ij} - \sum_{(x_j, x_i) \in \mathcal{A}} \sum_{x_i \in X} \psi_{ji} = \sum_{(x_i, x_j) \in \mathcal{A}} \sum_{x_i \in X} \psi_{ij} - \sum_{(x_i, x_j) \in \mathcal{A}} \sum_{x_j \in X} \psi_{ij} = \\ &= \sum_{(x_i, x_j) \in \mathcal{A}} \left(\sum_{x_i \in X} \psi_{ij} - \sum_{x_j \in X} \psi_{ij} \right) = 0. \end{aligned}$$

On the semi-flow's set, which can be defined in a network, we define the real nonnegative function :

$$\theta(\psi) = \begin{cases} \sum_{s_i > 0} s_i, & \text{if there exist at least an } s_i > 0, \\ 0, & \text{in a contrary case,} \end{cases}$$

which we shall call total *positive surpluss*. According to theorem 2, relation $\Phi(\psi) = 0$ in an extended network, implies the fact that it is a flow on the network. This property is evidently mutual.

We shall call *elementary semi-flow*, a semi-flow ψ' , thus any circuit of the network, which does not contain the arc (x_{n+1}, x_0) , has at least an arc to which corresponds a null value by the function ψ' . Similar to the property set forth by theorem 1, we have the following property, connected to the semi-flow notion

Theorem 3

Given a semi-flow ψ , on a network, of total positive surpluss $\theta(\psi)$, can be determine an elementary flow ψ' on the network, thus we have :

$$\theta(\psi') = \theta(\psi).$$

The demonstration of this property is simialr to that of theorem 1.

Lemma 3

Given a semi-flow ψ on a network, so that to the nod $x_i (x_i \neq x_{n+1})$ corresponds a surpluss $s_i > 0$, can be determined another semi-flow ψ' so that to the node x_i corresponds a null surpluss, without affecting the previous null surpluss, except the surpluss corresponding to the output. In addition, the two semi-flons verify the relation :

$$\theta(\psi') \leq \theta(\psi).$$

Demonstration.

According to theorem 3, we can suppose from the beginning that the semiflow ψ is elementary.

Because $s_i > 0$, there exists in the network at least one arc (x_i, x_j) , with $\psi_{ij} > 0$.

Therefore, we can replace the flow ψ by another, noted with $\psi^{(1)}$ which keeps unchanged the values associated to the arcs, except the value ψ_{ij} , associated to the arc (x_i, x_j) . which is replaced by :

$$\psi_{ij}^{(1)} = \psi_{ij} - \sigma$$

the quantity $\sigma > 0$ being for the time being undetermined. To the new semi-flow, $\psi^{(1)}$, shall correspond the surplusses of the nodes, equal to the old ones, except the following :

$$s_i^{(1)} = s_i - \sigma < s_i,$$

$$s_j^{(1)} = s_j + \sigma > s_j.$$

In case when $s_j \neq 0$, we consider :

$$\sigma = \min \{s_i; \psi_{ij}\}$$

and resume an operation similar to that from above, coming again to node x_i . In a contrary case, having initially $s_j = 0$, we obtain $s_j^{(1)} > 0$, which means that there exists at least one arc, (x_j, x_k) , with $\psi_{jk}^{(1)} = \psi_{jk} > 0$. We shall replace now also the semi-flow $\psi^{(1)}$ by another $\psi^{(2)}$, which keeps unchanges the values associated to the arcs, except for $\psi_{jk}^{(1)}$, which is replaced by :

$$\psi_{jk}^{(2)} = \psi_{jk}^{(1)} - \sigma = \psi_{jk} - \sigma.$$

Continuing in the same way, the given semi-flow being considered elementary, we shall necessariy come up, at one time, to use an arc which has as final extremity either the output from the network, or a node x_i wigt $s_i = 0$, cases in which we replace :

$$\sigma = \min \{\psi_{ij}, \psi_{jk}, \dots, \psi_{en+1}\}$$

respectively :

$$\sigma = \min \{\psi_{ij}, \psi_{jk}, \dots, \psi_{ei}\}.$$

Thus, the last semi-flow used shall keep all the initial null surplusses (eventually with the exception of the surpluss corresponding to the output), reducing the value of s_i with a quantity strictly positive.

Repeating the operations describes above, whenever it is necessary, we obtain the semi-flow ψ' , indicated by the lemma.

At last, watching the transformations of the surpluss positively total function, which occur at each modification of the semi-flow, we notice that we have :

$$\theta(\psi) \geq \theta(\psi^{(1)}) \geq \theta(\psi^{(2)}) \geq \dots$$

which justifies also the second part of the lemma's conclusion.

Corollary.

Given a semi-flow ψ , on a network, can be determined another semi-flow ψ' , so that :

$$\theta(\psi') = s'_{n+1} \leq \theta(\psi).$$

Observation :

Alike above, one can show that, starting from any semi-flow ψ , one can determine another semi-flow ψ' , to which corresponds a single negative surpluss in the network's source, and the positive total surpluss being also at least equal with the positive total surpluss of the given semi-flow.

Lemma 4

If on extended network can be define a normal semiflow ψ , with positiv total surpluss $\theta(\psi)$, then on the network can be defined also a flow φ , of total value $\bar{\Phi}$, so that there exists relation :

$$\bar{\Phi} \geq c_{n+1,0} - \theta(\psi).$$

Demonstration.

According to the corollary and to the observation from above, we can determine a semi-flow ψ , in which the only positive surplus is found in the output of the network, and the only negative surpluss in the source. Taking account of theorem 2, we have :

$$|\bar{s}_0| = \bar{s}_{n+1}$$

Such a semi-flow is evidently a flow on the network, of total value :

$$\bar{\Phi} = \bar{s}_{n+1}$$

Considering the extended network R' also on $\bar{\psi}$ like on a normal semi flow, we shall have thus the following relation :

$$\bar{\Phi} = c_{n+1,0} - \bar{s}_{n+1} = c_{n+1,0} - \theta(\bar{\psi}).$$

With these notations, according to the previous corollary, from the existence of any normal semi-flow ψ , can be deduced the existence of a flow $\bar{\varphi}$, of total value $\bar{\Phi}$, so that we have :

$$\bar{\Phi} = c_{n+1,0} - \theta(\bar{\varphi}) \geq c_{n+1,0} - \theta(\psi)$$

which must be demonstrated.

Theorem 4

Given an extended network R' and considering the set $\{\varphi\}$ of all the flows which can be defined on it, with the set of the total values $\{\Phi\}$, as well as the set $\{\psi\}$, of the normal semi-flows which can be defined on the network, we have :

$$\max \{\Phi\} = c_{n+1,0} - \min \{\theta(\psi)\}.$$

Demonstration

Denoting by φ^* a flow of total maximum value, Φ^* , which can be built-up on the network, we notice that one can attach to it a normal semi-flow ψ^* defined as follows :

$$\psi_{ij}^* = \begin{cases} c_{n+1,0}, & \text{for } (x_i, x_j) \neq (x_{n+1}, x_0), \\ \varphi_{ij}, & \text{in the contrary case.} \end{cases}$$

The surplus function, corresponding to this semi-flow, is the following :

$$s_i = \begin{cases} c_{n+1,0} - \Phi^*, & \text{for } x_i \equiv x_{n+1}, \\ \Phi^* - c_{n+1,0}, & \text{for } x_i \equiv x_0, \\ 0, & \text{in the contrary case.} \end{cases}$$

Because necessarily we have $\Phi^* \leq c_{n+1,0}$, it results that for this semi-flow takes place the following repartition :

$$\Phi(\psi^*) = c_{n+1,0} - \Phi^* \quad (6)$$

On the other side, by lemma 4 was shown that, given any normal semi-flow ψ , to it corresponds a flow on the network, of total value Φ , o that :

$$\Phi \geq c_{n+1,0} - \theta(\psi).$$

Considering as semi-flow ψ , a normal semi-flow, to which corresponds a minimum value of the positive total surplus function and noting with $\bar{\Phi}$ the total flow corresponding to it by lemma 4, we have :

$$\bar{\Phi} \geq c_{n+1,0} - \min \{\theta(\psi)\},$$

or :

$$\min \{ \theta(\psi) \} \geq c_{n+1,0} - \bar{\Phi}.$$

Taking account of the fact that $\bar{\Phi} < \Phi^*$, this relation becomes :

$$\min \{ \theta(\psi) \} \geq c_{n+1,0} - \Phi^* = \theta(\psi^*).$$

From this results that :

$$\theta(\psi^*) = \min \{ \theta(\psi) \},$$

a fact which replaced in relation (6), gives :

$$\min \{ \theta(\psi) \} = c_{n+1,0} - \Phi^*,$$

identical equality to the relation which had to be demonstrated.

The theorem from above allows the replacing of the determination problem of the maximal flow in the network, with that of the determination of a semi-flow of positive total minimum surpluss. Once such a normal surpluss determined with a positive total minimum surpluss, transferring all its surplusses in the two extremities (the positive one in the output and the negative one in the source), is obtained a flow on the network, with total maximum value.

Description and justification of the algorithm.

Starting from the idea to determine a normal semi-flow with positive total minimum surpluss, having initially any normal semi-flow. we shall try to transfer the positive surplusses to the negative ones so that part of the positive total surplusses (or entirely) should compensate, without affecting the value attached to arc (x_{n+1}, x_0) . These transfers of surpluss are effectuated as it is shown in continuation.

Given a normal elementary semi-flow ψ , we consider the surplusses corresponding to the nodes, following which we use the two rules of transfer of a positive surpluss from a node x_i , in another x_j adjoint (tied to it by an arc) :

a) If the network contains the arc $(x_i, x_j) \neq (x_{n+1}, x_0)$, for which $\psi_{ij} > 0$, we replace the semi-flow ψ by another, ψ' , which keeps the old values attached to the arcs, except the one brought about, which is replaced by :

$$\psi'_{ij} = \psi_{ij} - \min \{ s_i ; \psi_{ij} \}.$$

In this way, the whole positive surpluss, corresponding [to node x_i , or only a part of it, is transferred to node x_j -

b) If the network contains the arc (x_j, x_i) , for which $\psi_{ji} < c_{ji}$, we replace the semi-flow ψ by another, ψ' , which keeps the old values attached to the arcs, except that in cause, which is replaced by :

$$\psi'_{ji} = \psi_{ji} + \min \{ s_i ; c_{ji} - \psi_{ji} \}.$$

In this way, the whole positive surpluss of node x_i , or part of it, is transferred in node x_j .

The practical application of the two rules of transfer from above, can be done by a marking process of the nodes on the network, starting from the nodes with positive surpluss, as follows :

- we mark first, all nodes with positive surpluss ;
- we mark any other node from the network, in which can be transferred, by the above rules, a part or the whole positive surpluss found in one of the marked nodes ;
- the process of marking the nodes continues until a node with negative surpluss was marked, or until this was not realized, thus it is no more possible to mark any other node of the network.

In case that by the process of marking indicated, was marked a node with negative surpluss, is determined, with the aid of the marks made, a chain which joins a node with positive surpluss, with a node which has a negative surpluss, on all the edges of the chain being possible the transferr of a positive surpluss. Along this chain is made the transfer of a part, or of the whole surpluss corresponding to the initial node of the chain (preferably is transferred as much as possible from this positive surpluss). Following this transfer, is obtained another normal semi-flow on the network, with a positive total surpluss smaller than the previous one. Continuing, the marks made are wiped out and the above operations are repeated, using the new semi-flow.

In case that, by the marking process indicate, one cannot mark any node wiht negative surpluss, there was obtained a normal semi-flow with positive total minimum surpluss, according to the theorem that follows.

Theorem 5

Given a normal semi-flow ψ , defined on an extended network, if by the marking process of the nodes one cannot mark any negative surpluss node, tham the semi-flow ψ , corresponds a posstiiva total minimum.

Demonstration

Given a normal semi-flow ψ for which we proceed by marking, the nodes, by which not a single nod with negative surpluss could be marked. According to lemma 3, the output from the network is one of the marked nodes. Alike, the network's source is a node which cannot be marked.

Denoting by P the set of nodes which cannot be marked, partition $(P, \bar{P})$ of the network's nodes, it forms a cut S . Taking account of the two rules of applied marking, we have :

$$\sum_{\substack{x_i \in P \\ x_j \in \bar{P}}} \psi_{ij} = \sum_{\substack{x_i \in P \\ x_j \in \bar{P}}} c_{ij} = C(S),$$

and :

$$\psi_{ij} = 0, \text{ if } x_i \in P \text{ and } x_j \in \bar{P}.$$

Transferring all the positive surplusses in output and the negative ones in the network's source, is obtained a flow φ on the network, so that :

$\varphi_{ij} = \psi_{ij}$, whatever would be $x_i \in P$ and $x_j \in \bar{P}$, or inverse.

In addition, we have :

$$\theta(\varphi) = \theta(\psi).$$

The total flow, going through the indicated cut, will be thus :

$$\Phi = \sum_{\substack{x_i \in P \\ x_j \in \bar{P}}} \varphi_{ij} - \sum_{\substack{x_i \in \bar{P} \\ x_j \in P}} \varphi_{ij} = \sum_{\substack{x_i \in P \\ x_j \in \bar{P}}} \psi_{ij} = C(S).$$

This flow, being equal with the capacity some of a cut from the network according to Fulkerson's theorem, has maximal value, which brings the conclusion that :

$$\theta(\psi) = \Phi(\varphi).$$

has minimal values. The theorem is thus demonstrated.

Regarding the initial determination of a normal elementary semi-flow, in an extended network, this can be realized in different ways. If we shall consider initially as semi-flow ψ the following :

$$\psi_{ij} = \begin{cases} 0, & \text{for } (x_i, x_j) \neq (x_{n+1,0}, x_0), \\ c_{n+1,0}, & \text{for } (x_i, x_j) = (x_{n+1,1}, x_0), \end{cases}$$

and we shall proceed, similarly to the method exposed, to the transfer of the negative surpluss from the network's source, at the output of the network, the algorithm coincides with the Ford-Fulkerson's algorithm.

Often, a normal elementary semi-flow, which we take as starting point, results from a previous solution of another flow problem in the network. Following, we propose as normal elementary initial semi-flow on the extended network, *maximal semi-flow* which we can define in the following way :

$$\psi_{ij} = c_{ij},$$

for all the arcs of the extended network. Because this semi-flow could be not elementary (which happens in a simple network with circuits) one looks successively for a circuit which is of no use to the arc (x_{n+1}, x_0) and which has all the arcs with values ψ_{ij} attaches strictly positive and one deduces the smallest value from all these values, until such circuits do not exist any longer. The practical determination of some circuits of this kind, on a big network, can be done with the help of a marking process, starting the marking with any node x_i , we mark with x_i all the nodes x_j , for which the network contains the arc $(x_i, x_j) \neq (x_{n+1}, x_0)$,

with $\psi_{ij} > 0$, continuing in the same way with the marked nodes; this process continues until was marked a node which has been marked before, the circuit being determined then with the aid of the effectuated marks.

If this marking process cannot continue any more, without a node being marked twice, the marking process is revived, starting from another node, nonmarked in the previous process and so on, until all the nodes of the network have been marked.

Observations

1. The rules of transfer of the positive surpluss, indicated above can be used also for the cumulation of some positive surplusses in less nodes of the network.

2. Alike with the transfer rules of the positive surpluss between the neighbouring nodes, can be deduced the transfer laws of the negative surpluss, which can be used also in the case of the final transfer of the whole negative surpluss in the network's source.

3. The determination operations of an optimal flow can be applied starting also from a normal nonelementary semi-flow, the result could be an optimal nonelementary flow. In this case, starting from the last on, one determines at the end of the algorithm an optimal elementary flow by the same procedure as above.

Transposure of the algorithm on the matrix

The algorithm described can be applied mor easily, in the case of a network of big dimensions, if all its corresponding operations are effectuated on a table, as it is shown in continuation.

Step 0 (determination of a normal initial semi-flow). Given an extended network R' , is considered a matrix $\psi = \|\psi_{ij}\|$, $i, j = 0, 1, 2, \dots, n+1$, whose elements are defined thus:

$$\psi_{ij} = \begin{cases} c_{ij}, & \text{if the network contains the arc } (x_i, x_j) \\ 0, & \text{in the contrary case.} \end{cases}$$

The elements, different of zero, of this matrix are written in the corresponding cells of a quadrativ table with $(n+2)$ lines and $(n+2)$ columns the other cells remaining free. In addition, for every node x_i , in cell (x_i, x_i) , on the principal diagonal of the table, is written the quantity:

$$s_i = \sum_{j=0}^{n+1} c_{ij} - \sum_{j=0}^{n+1} c_{ji}.$$

Practically, this quantity is obtained deducing from the total sum the previous elements of line x_i , the sum of all previous elements on column x_i . According to lemma 2, the sum of all elements of the principal diagonal of the table is equal to zero.

Step 1 (determination of a normal elementary semi-flow). On the other hand on the table built-up above, is made the marking of columns, as follows :

- is marked first the column x_0 with an asterix,
- is marked with x_0 all columns x_j , for which $\psi_{0j} > 0$;
- for each column x_j , marked like above, is marked with x_j all columns x_k , for which $\psi_{jk} > 0$, but never using $\psi_{1,0}$;
- one continues marking the nodes in the same way, until one can mark a column which was still marked, or until this fact has not taken place, although the marking process cannot continue any more.

From the first situation above, starting in opposite sense, from the node corresponding to the column marked twice, with the help of marks, a circuit is determined. From the values ψ_i , attached to all arcs of the circuit, we diminish with the smallest of them and we take again the marking procedure of the columns, starting with column x_0 .

In the second situation from above, we say that the previous markings ended and we take again the method of marking the columns, starting with a column which was not marked in any series concluded before. These operations continue as long as they are possible.

Step 2 (determination of a normal elementary semi-flow, with positive totally minimum surplus). The positive surpluses are transferred from a cell into another cell in the principal diagonal, using the following two rules :

(a) Is transferred the maximal possible quantity of positive surplus from cell (x_i, x_i) in the cell (x_j, x_j) , deducing this quantity from the semi-flow corresponding to the cell (x_i, x_i) .

(b) Is transferred the maximal possible quantity of positive surplus, from cell (x_i, x_i) in the cell (x_j, x_j) , adding this quantity to the semi-flow corresponding to the cell (x_j, x_i) .

Using adequately the two transfer rules, from above, we can reach the following result : given the compensation of a part from the positive total surplus with the negative one, or the cumulation of the positive total surplus in less nodes.

Using practically the two rules, shown above, can be done also by a marking procedure of the columns in the table used, such as :

- marking all the columns with positive surplus with an asterix
- if the column x_i is marked, we mark with $-x_i$ all the columns x_j , for which $\psi_{ij} > 0$ and with $+x_i$ all the columns x_k , for which $\psi_{ik} < e_{ik}$, not using ever for this the value $\psi_{i+1,0}$;
- marking continues until has been marked a column x_s , to which corresponds a negative surplus, or until this fact has not happened, though marking cannot continue any longer.

In the first case from above, markings taking forth the determination of a succession of cells from the table, in which must be modified the values of the semi-flow, such as a part (of all) of the positive surplus from the cell (x_i, x_i) should compensate with a part (or all) of the negative sur-

pluss from the cell (x_s, x_e) . This part, which we denote by σ , is determined taking account that the new semi-flow ψ , should verify the relations:

$$0 \leq \psi'_{ij} \leq c_{ij}, \text{ whatever would be } (x_i, x_j) \in A.$$

Given cell (x_u, x_v) from the mentioned succession: if column x_v was marked with $-x_u$, we shall deduce the quantity σ from the semi-flow ψ_{uv} , and if the column x_v was marked with $+x_u$, we shall add the quantity σ to the semi-flow ψ_{uv} , the result being a compensation of σ positive surpluss units from the node x_i , with σ negative surpluss units from the node x_e . Then, the marking process is taken again.

In case of the second situation from above, we deduce that the normal semi-flow obtained has positive total minimum surpluss, being at the same time also elementary.

Step 3 (determination of an elementary flow of total minimum value). Starting from the semi-flow obtained previously, we transfer all the positive surplusses in the out-put of the network, and all the negative surplusses in the network's source. For instance, for the transfer of the positive surplusses in the output, we can determine, with the aid of the marking procedure of the columns, indicated before, a succession of cells which permit the transfer of positive surplusses in the cell (x_{n+1}, x_{n+1}) , these successions being used in step 2. Also, by a similar marking procedure of the table's columns, which can easily be deduced, are determined successions of cells of the table, which permit the transfer of negative surplusses in cell (x_0, x_0) .

Finally, is obtained a flow of total maximal value in the simple network. Denoting with s_{n+1}^* the total positive surpluss, transferred in the output of the network like above, and the putting:

$$\psi_{n+1,0} = c_{n+1,0} - s_{n+1}^*,$$

the final semi-flow ψ becomes an optimum flow on the extended network.

Numerical applications

1. In the network of fig. 1, let us determine a maximal flow, applying the described algorithm.

The matrix from fig. 2, contains the initial semi-flow as well as the quantities s_i , $i = 0, 1, \dots, 5$. The matrices from figures 3, 4, 5 and 6 contain the respective semi-flow $\psi_1, \psi_2, \psi_3, \psi_4$.

In the matrix from fig. 6, was applied the marking operation, as well as the transfer of semi-flows, remaining, and in the matrix of fig. 7 was obtained the maximal flow needed.

2. We admit that in the network 1, was modified the capacity $c_{13} = 9$, becoming $\tilde{c}_{13} = 7$. The correction made to the maximal flow in figure 7, for the new situation is realized in the matrix of figure 8.

3. We admit that in the network from fig. 1 the output is in node x_3 , and the source remains in node x_0 . On the matrix of figure 9 is determined the new maximal flow, starting from the old optimum flow from figure 7.

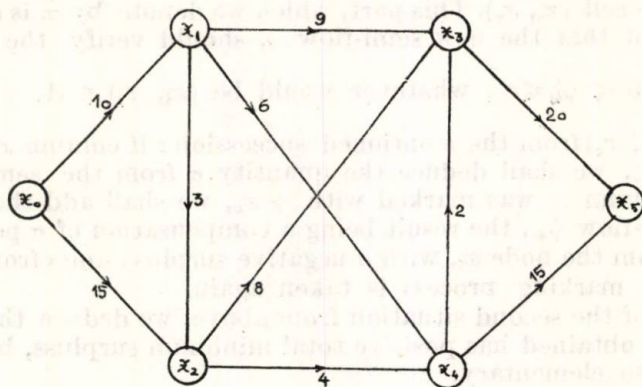


Fig. 1

ψ_{10}, Δ

x_0	x_1	x_2	x_3	x_4	x_5
x_0	0	10	15		
x_1		8	3	9	6
x_2			-6	8	4
x_3				1	20
x_4				2	7
x_5					15

Fig. 2

ψ_{11}

x_0	x_1	x_2	x_3	x_4	x_5
x_0	0	10	15		
x_1		5	0*	9	6
x_2			-3	8	4
x_3				1	20
x_4				2	7
x_5					15

Fig. 3

ψ_{12}

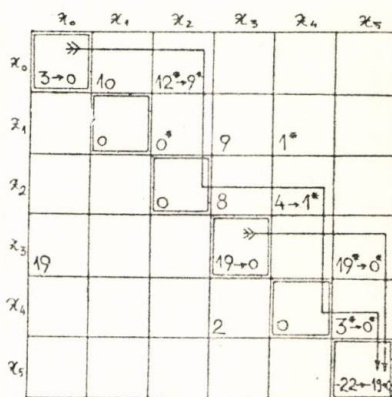
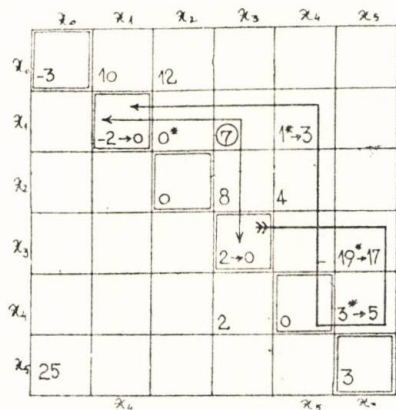
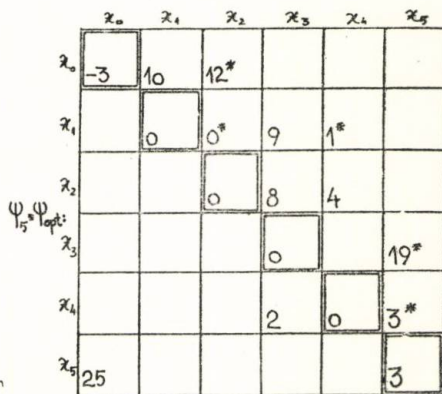
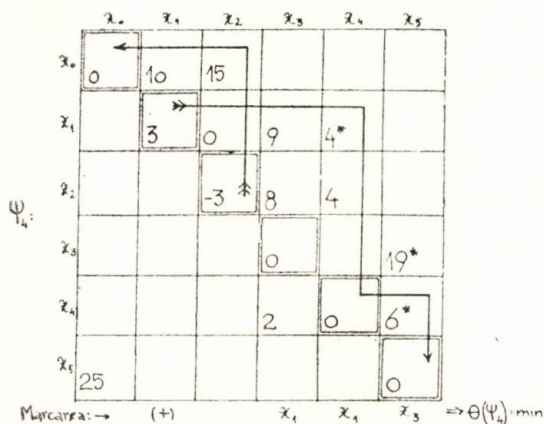
x_0	x_1	x_2	x_3	x_4	x_5
x_0	0	10	15		
x_1		5	0*	9	6
x_2			-3	8	4
x_3				0	19
x_4				2	7
x_5	25				15

Fig. 4

ψ_{13}

x_0	x_1	x_2	x_3	x_4	x_5
x_0	0	10	15		
x_1		5	0*	9	6
x_2			-3	8	4
x_3				0	19*
x_4				2	0
x_5	25				8*

Fig. 5



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КРАЕВАЯ ЗАДАЧА С ПАРАМЕТРОМ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

М. М. КОНСТАНТИНОВ, Д. Д. БАЙНОВ

1. Постановка задачи, основные предположения

В настоящей работе находятся условия разрешимости одного класса краевых задач для дифференциальных уравнений с отклоняющимися аргументом и параметром. Задачи этого типа рассматривались также в [1]. Отметим, что результаты работы [1] можно обобщить и усилить.

Рассмотрим задачу об определении параметра λ из некоторого компакта Λ , так чтобы выполнялось краевое условие

$\alpha\lambda + \Phi(\lambda, y(T)) = 0$, $T > 0$ для решения $y = y(s)$ начальной задачи

$$\begin{aligned} y'(s) &= Y(s, y(s), y(\tau(s, \lambda)), \lambda), \quad s > 0 \quad (\tau \leq s) \\ y(s) &= \psi(s, \lambda), \quad s \leq 0. \end{aligned} \tag{1}$$

Положим

$$\begin{aligned} y(s) &= x(s) + \psi(s, \lambda), \quad s \leq 0; \\ y(s) &= x(s) + \psi(0, \lambda), \quad s > 0; \quad s = Tt. \end{aligned}$$

Тогда рассматриваемая задача сводится к задаче: Найти $\lambda \in \Lambda = [\lambda_1, \lambda_2]$, так чтобы $\alpha\lambda + F(\lambda, x(1)) = 0$, где $x = x(t)$ — решение начальной задачи

$$\begin{aligned} \dot{x}(t) &= X(t, x(t), x(\Delta(t, \lambda)), \lambda), \quad t \in I = [0, 1] \\ x(t) &= 0, \quad t \leq 0 \end{aligned} \tag{2}$$

(функции X , Δ , и F получены из элементов начальной задачи (1) при помощи сделанной подстановки).

Предложим, что выполнены следующие условия (A):

A1. Функции $X(t, \xi_1, \xi_2, \lambda)$, $\Delta(t, \lambda)$ и $F(\lambda, \xi)$ определены непрерывны по t в областях $Q = I \times D^2 \times \Lambda$, $Q_1 = I \times \Lambda$ и $Q_2 = \Lambda \times D$, где $D = \{\xi : |\xi| \leq d\}$.

A2. В областях Q , Q_1 и Q_2 функции X , Δ и F удовлетворяют условиям Липшица

$$\begin{aligned} |X(t, \xi_1, \xi_2, \lambda) - X(t, \bar{\xi}_1, \bar{\xi}_2, \bar{\lambda})| &\leq M|\lambda - \bar{\lambda}| + \sum_{i=1}^2 \xi L_i |\xi_i - \bar{\xi}_i|, \\ |\Delta(t, \lambda) - \Delta(t, \bar{\lambda})| &\leq \mu_1 |\lambda - \bar{\lambda}|, \\ |F(\lambda, \xi) - F(\bar{\lambda}, \bar{\xi})| &\leq L|\xi - \bar{\xi}| + \mu_2 |\lambda - \bar{\lambda}|. \end{aligned}$$

А3. Имеют место неравенства $d_0 \leq d$ и $\Delta_0 \geq 0$, где

$$d_0 = \int_0^1 X_0(t) dt,$$

$$\Delta_0 = \max \{t - \Delta(t, \lambda) : (t, \lambda) \in Q_1\},$$

$$X_0(t) = \sup \{|X(t, \xi_1, \xi_2, \lambda)| : (\xi_1, \xi_2, \lambda) \in D^2 \times \Lambda\}.$$

2. Существование и единственность решений краевой задачи

Рассмотрим множество $\Omega = \omega \times \Lambda$, где ω — множество всех функций x на интервале $[\Delta_1, 1]$, $\Delta_1 = \min \{\Delta(t, \lambda) : (t, \lambda) \in Q_1\}$ таких, что $x(t) = 0$, $t \leq 0$ и

$$|x(t)| \leq \int_0^t X_0(s) ds, \quad t \in I;$$

$$|x(t_1) - x(t_2)| \leq m|t_1 - t_2|, \quad t_1, t_2 \in I,$$

где $m = \max \{X_0(t) : t \in I\}$.

Пусть оператор Π определен на Ω формулой $\Pi z = (\Pi x, \Pi \lambda)$, $z = (x, \lambda)$:

$$\Pi x(t) = \int_0^t X(s, x(s), x(\Delta(s, \lambda)), \lambda) ds, \quad t \in I;$$

$$\Pi x(t) = 0, \quad t \leq 0$$

$$\Pi \lambda = -\alpha^{-1} F(\lambda, x(1)).$$

Для существования хотя бы одного решения операторного уравнения $z = \Pi z$ (это уравнение эквивалентно начальной задаче (2) с краевым условием $\alpha \lambda + F(\lambda, x(1)) = 0$. достаточно выполнение условий (А) и условия

$$-\alpha^{-1} F(\lambda, x(1)) \in \Lambda$$

для всех $(x, \lambda) \in \Omega$. В этом случае будет иметь место включение $\Pi \Omega \subset \Omega$, откуда в силу принципа Шаудера о неподвижной точке следует существование решения краевой задачи.

Для единственности решения надо потребовать дополнительные ограничения.

ТЕОРЕМА. Пусть выполнены условия (А) и (З), Пусть кроме того $L_1 + L_2 < 1$ и

$$|\alpha| > \mu_2 + \frac{L(M + \mu_1 m L_2)}{1 - L_1 - L_2}$$

Тогда краевая задача имеет единственное решение.

Доказательство. Введем в Ω норму

$$\|Z\|_{\Omega} = \|x\| + q|\lambda|,$$

где $\|x\| = \max \{|x(t)| : t \in I\}$, а число $q > 0$ определим ниже.

Для $z, \bar{z} \in \Omega$ ($\bar{z} = (x, \bar{\lambda})$) имеем

$$\|Px - P\bar{x}\| \leq (L_1 + L_2)\|x - \bar{x}\| + (M + \mu_1 m L_2)|\lambda - \bar{\lambda}|$$

и

$$|P\lambda - P\bar{\lambda}| \leq |\alpha|^{-1}(L\|x - \bar{x}\| + \mu_2|\lambda - \bar{\lambda}|).$$

С другой стороны в силу определения числа $|\alpha|$, число q можно выбрать таким образом, чтобы

$$L_1 + L_2 + qL|\alpha|^{-1} < 1$$

и

$$M + \mu_1 m L_2 + q\mu_2|\alpha|^{-1} < q.$$

Теперь ссылка на принцип Банаха о неподвижной точке завершает доказательство теоремы.

*Высший Машинно — Электротехнический
Институт Имени В. И. Ленина — София*

*Пловдивский Университет
Имени П. Хилендарского*

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Тогда для любых $\epsilon > 0$ найдутся $\delta > 0$ и $N > 0$ такие, что

для любых $n > N$ и $k > 0$ имеем

$$|f_n(x) - f(x)| < \epsilon$$

для любых $x \in E$ и $n > N$. Тогда для любых $\epsilon > 0$ найдутся $\delta > 0$ и $N > 0$ такие, что

для любых $n > N$ и $k > 0$ имеем

$$|f_n(x) - f(x)| < \epsilon$$

$$|f_n(x) - f(x)| < \epsilon$$

С другой стороны в силу непрерывности функции f на E можно выбрать такие образы U и V , что

$$U \cap V = \emptyset$$

$$U \cap V = \emptyset$$

Теперь остается на основании леммы о непрерывности функции доказать, что

Вспомогательная лемма. Пусть f — непрерывная функция на E . Тогда для любых $\epsilon > 0$ найдутся $\delta > 0$ и $N > 0$ такие, что для любых $n > N$ и $k > 0$ имеем

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A METHOD FOR FINDING THE KERNEL OF A DIGRAPH

DĂNUȚ MARCU

The problem of finding the kernel of a digraph has been discussed extensively in the literature [1], [3], [4], [5], [6].

In [4] Richardson gives a constructive theorem for the existence of a kernel in a digraph without odd circuits.

Our paper suggests a new and simple method for finding the kernel of a digraph without odd circuits, a method for which we can write a simple and efficient computer program.

Moreover, our algorithm can be considered as a simplified demonstration of Richardson's theorem.

Let $D = \langle \mathcal{N}, \mathcal{A} \rangle$ be a digraph [2] with $\mathcal{N} = \{n_1, n_2, \dots, n_p\}$ the set of nodes, and $\mathcal{A}$ the set of arcs. If certain members of $\mathcal{A}$ can be placed in a sequence of the form $\langle n_{i_1}, n_{i_2} \rangle, \langle n_{i_2}, n_{i_3} \rangle, \dots, \langle n_{i_{s-1}}, n_{i_s} \rangle$, then the set $\{\langle n_{i_1}, n_{i_2} \rangle, \langle n_{i_2}, n_{i_3} \rangle, \dots, \langle n_{i_{s-1}}, n_{i_s} \rangle\}$ is a path of length $s - 1$ from n_{i_1} to n_{i_s} , a path denoted by $\gamma[n_{i_1}, n_{i_s}]$. The path is a circuit, if $n_{i_1} = n_{i_s}$. A digraph is strongly connected if and only if for every two nodes n and m , there exist the paths $\gamma[n, m]$ and $\gamma[m, n]$. Let n be an arbitrary node of $\mathcal{N}$. We denote by $\mathcal{N}_{[n]}^+$ the set: $\mathcal{N}_{[n]}^+ = \{n\} \cup \{m \in \mathcal{N} \mid \text{there exists } \langle n, m \rangle \in \mathcal{A}\}$.

A subset $\mathcal{K} \subset \mathcal{N}$ is a kernel [5] for the digraph D , if and only if:

$$(\alpha)^+ \mathcal{N} [\mathcal{K}] = \mathcal{N}.$$

(β) for every two nodes $n, m \in \mathcal{K}$ we have $\langle n, m \rangle \notin \mathcal{A}$ and $\langle m, n \rangle \notin \mathcal{A}$, where $\mathcal{N}^+[\mathcal{K}] = \bigcup_{k \in \mathcal{K}} \mathcal{N}_{[k]}^+ \supset \mathcal{K}$.

Let $D = \langle \mathcal{N}, \mathcal{A} \rangle$ be a strongly connected digraph without odd circuits, and n_0 an arbitrary node of $\mathcal{N}$; we consider the sets $\mathcal{N}_0 = \{m \in \mathcal{N} \mid \text{there exists } \gamma[n_0, m], \text{ and the length of } \gamma[n_0, m] \text{ is odd}\}$, and $\overline{\mathcal{N}}_0 = \mathcal{N} \setminus \mathcal{N}_0$.

Proposition 1

The set $\overline{\mathcal{N}}_0$ is a kernel of the digraph D .

Proof. Let $\langle n^*, m^* \rangle$ be an arbitrary arc of $\mathcal{A}$. Suppose that the nodes n^* and m^* belong to $\overline{\mathcal{N}}_0$. Because D is strongly connected without odd circuits, and taking into account the construction of $\mathcal{N}_0$, there exist the even paths $\gamma[n_0, n^*]$, $\gamma[m^*, n_0]$, that implies the existence of the odd circuit $[n_0, \dots, n^*, m^*, \dots, n_0]$; contradiction. Hence, D does not contain arcs $\langle n^*, m^* \rangle$ with $n^*, m^* \in \overline{\mathcal{N}}_0$. Similarly we show that D does not contain arcs $\langle n^*, m^* \rangle$ with $n^*, m^* \in \mathcal{N}_0$. At result, for all arcs $\langle n^*, m^* \rangle$ of $\mathcal{A}$, we have: $n^* \in \mathcal{N}_0$ and $m^* \in \overline{\mathcal{N}}_0$, or, $n^* \in \overline{\mathcal{N}}_0$ and $m^* \in \mathcal{N}_0$.

Let now m_0 be an arbitrary node of $\mathcal{N}_0 = \mathcal{N} \setminus \overline{\mathcal{N}}_0$ and $\overline{m}_0$ an arbitrary node of $\overline{\mathcal{N}}_0 = \mathcal{N} / \mathcal{N}_0$. Because D is strongly connected there exists at

least a path $\gamma[\bar{m}_0, m_0] = [\bar{m}_0, t_1, t_2, \dots, t_h, m_0]$, and for the arc $\langle t_h, m_0 \rangle$ we have $t_h \in \mathcal{N}_0$. Hence, for every $m_0 \in \mathcal{N}_0$ there exists $t_0 \in \mathcal{N}_0$ so that $\langle t_0, m_0 \rangle \in \mathcal{A}$. It results $\mathcal{N}^+[\mathcal{N}_0] = \mathcal{N}$, and by the previous results the set $\mathcal{N}_0$ is a kernel of the digraph D . (Q.E.D.).

Let $\mathfrak{C}$ be a subset of $\mathcal{N}$. We denote by $\text{id}[\mathfrak{C}]$ the number: $\text{id}[\mathfrak{C}] = |\{a \in \mathcal{A} \mid a = \langle n, m \rangle, n \notin \mathfrak{C}, m \in \mathfrak{C}\}|$.

Let now $D = \langle \mathcal{N}, \mathcal{A} \rangle$ be a digraph without odd circuits. From [5] it results that D contains at least a kernel. Our following algorithm suggests a method for finding it:

1. $i := 0$; Consider the digraph $D^{(0)} = \langle \mathcal{N}^{(0)}, \mathcal{A}^{(0)} \rangle$ where

$$\mathcal{N}^{(0)} = \mathcal{N} \text{ and } \mathcal{A}^{(0)} = \mathcal{A}.$$

2. Let $\mathfrak{C}^{(i)}$ be a strongly connected component of

$$D^{(i)}, \text{ with } \text{id}[\mathfrak{C}^{(i)}] = 0.$$

For the digraph $\langle \mathfrak{C}^{(i)}, [\mathfrak{C}^{(i)} \times \mathfrak{C}^{(i)}] \cap \mathcal{A}^{(i)} \rangle$ we can construct a kernel $\mathcal{K}^{(i)} \subset \mathfrak{C}^{(i)}$ by following the proposition 1.

3. $i := i + 1$; construct the digraph $D^{(i)} = \langle \mathcal{N}^{(i)}, \mathcal{A}^{(i)} \rangle$, where $\mathcal{N}^{(i)} \setminus \mathcal{N}^{(i-1)} = \mathcal{N}^+[\mathcal{K}^{(i-1)}]$, and

$$\mathcal{A}^{(i)} = [\mathcal{N}^{(i)} \times \mathcal{N}^{(i)}] \cap \mathcal{A}^{(0)}$$

4. If $\mathcal{N}^{(i)} = \emptyset$, STOP; else go to 2.

Let r be the value of i which $\mathcal{N}^{(r)} = \emptyset$.

Proposition 2

The set $\mathcal{K} = \bigcup_{i=0}^{r-1} \mathcal{K}^{(i)}$ is a kernel of the digraph D .

Proof. From the construction of $\mathcal{N}^{(i)}$, it results

$$\mathcal{N}^+[\mathcal{K} = \bigcup_{i=0}^{r-1} \mathcal{K}^{(i)}] = \mathcal{N}.$$

Now, we must show that there is no arc $\langle x, y \rangle$ with $x \in \mathcal{K}^{(i)}$ and $y \in \mathcal{K}^{(j)}$, for all $i, j \in \{1, 2, \dots, r-1\}$. If $i = j$ or $i < j$, this fact results from the construction of $\mathcal{K}^{(i)}$.

Let $i > j$ and suppose that there exists an arc $\langle x, y \rangle$ with $x \in \mathcal{K}^{(i)}$ and $y \in \mathcal{K}^{(j)}$. Because $i > j$, we have

$$x \in \mathcal{N}^{(s)}, \text{ for all } s = j, j+1, \dots, i \quad (1)$$

In the digraph $D^{(j)}$ if $x \notin \mathfrak{C}^{(j)}$, then $\text{id}[\mathfrak{C}^{(j)}] \neq 0$, and we have a contradiction. (see the step 2 of the algorithm).

Hence, $x \in \mathfrak{C}^{(j)}$. Because $\mathcal{K}^{(j)}$ is a kernel of the digraph $\langle \mathfrak{C}^{(j)}, [\mathfrak{C}^{(j)} \times \mathfrak{C}^{(j)}] \cap \mathcal{A}^{(0)} \rangle$, and $x \in \mathfrak{C}^{(j)}$, we have $x \in \mathcal{N}^+[\mathcal{K}^{(j)}]$, that implies, by the construction of $\mathcal{N}^{(i)}$, the relation:

$x \notin \mathcal{K}^{(j-1)}$, which contradicts (1).
Hence, there is no arc $\langle x, y \rangle$ with $x \in \mathcal{K}^{(i)}$ and $y \in \mathcal{K}^{(j)}$, for all $i, j \in \{1, 2, \dots, r-1\}$ i.e., $\mathcal{K} = \bigcup_{i=0}^{r-1} \mathcal{K}^{(i)}$ is a kernel of D , as required.

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Str. Piața M. Kogălniceanu 8,
 Sc. B, et. 4, a . 22, Sector 6,
 Bucharest

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THE STATE OF NEW YORK
IN SENATE
January 10, 1900.
REPORT
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MAY 1, 1899.

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FOR ITS CONSIDERATION.

UN ESPACE ISOMÉTRIQUE À L'ESPACE PAUL LÉVY DES FONCTIONS DE RÉPARTITION

N. MICU et F. NIȚULESCU

Définition.

L'application $F^* : R \rightarrow \bar{R}$ est une fonction de répartition inverse si elle satisfait les conditions suivantes :

- a) F^* est finie dans l'intervalle $(0,1)$, $F^*(t) = -\infty$ si $t < 0$, $F^*(t) = +\infty$ si $t \geq 1$;
- b) F^* est non décroissante sur R ;
- c) $\lim_{t \nearrow t_0} F^*(t) = F^*(t_0)$ ($\forall$) $t_0 \in R$.

Pour la suite nous noterons par $\mathcal{F}^*$ la famille des fonctions de répartition inverses.

Considérons l'application $\rho^* : \mathcal{F}^* \times \mathcal{F}^* \rightarrow R$ où pour toute F_1^* et F_2^* de $\mathcal{F}^*$ on a :

$$\rho^*(F_1^*, F_2^*) = \inf \{ \varepsilon : F_1^*(t - \varepsilon) - \varepsilon \leq F_2^*(t) \leq F_1^*(t + \varepsilon) + \varepsilon, t \in (0, 1) \}.$$

Il est facile à remarquer que pour toute F_1^* et F_2^* de $\mathcal{F}^*$ ou a : $0 \leq \rho^*(F_1^*, F_2^*) \leq 1$, et si les inégalités

$$F_1^*(t - \varepsilon) - \varepsilon < F_2^*(t) \leq F_1^*(t + \varepsilon) + \varepsilon$$

sont vérifiées pour tout $t \in (0, 1)$, alors elles sont vérifiées pour tout $t \in R$

Remarque

Dans la définition de ρ^* le infimum est atteint.

En effet, si $\varepsilon_0 = \rho^*(F_1^*, F_2^*)$ et $\varepsilon_n \downarrow \varepsilon_0$ où

$$\varepsilon_n \in \{ \varepsilon : F_1^*(t - \varepsilon) - \varepsilon \leq F_2^*(t) \leq F_1^*(t + \varepsilon) + \varepsilon, t \in (0, 1) \}$$

alors pour $n \rightarrow +\infty$, de la relation

$$F_1^*(t) \leq F_1^*(t + \varepsilon_n) + \varepsilon_n \quad t \in (0, 1)$$

résulte

$$F_2^*(t) \leq F_1^*(t + \varepsilon_0) + \varepsilon_0 \quad t \in (0, 1)$$

En remplaçant t par $t + \varepsilon_n - \varepsilon_0$ dans la relation

$$F_1^*(t - \varepsilon_n) - \varepsilon_n \leq F_2^*(t) \quad t \in (0, 1)$$

on obtient

$$F_1^*(t - \varepsilon_0) - \varepsilon_n \leq F_2^*(t + \varepsilon_n - \varepsilon_0) \quad t \in (0, 1)$$

et pour $n \rightarrow \infty$, il en résulte :

$$F_1^*(t - \varepsilon_0) - \varepsilon_0 \leq F_2^*(t) \quad t \in (0, 1)$$

Proposition 1

ρ^* est une distance sur $\mathcal{F}^*$.

Démonstration

Il est évident que $\rho^*(F^*, F^*) = 0$ pour toute $F^* \in \mathcal{F}^*$.
Réciproquement, si $\rho^*(F_1^*, F_2^*) = 0$, alors, d'après la remarque précédente, on a $F_1^*(t) = F_2^*(t)$ pour toute $t \in (0, 1)$, donc pour tout $t \in R$.

Si

$$F_1^*(t - \varepsilon) - \varepsilon \leq F_2^*(t) \quad (\forall) t \in R$$

alors en remplaçant t par $t + \varepsilon$, il en résulte

$$F_1^*(t) \leq F_2^*(t + \varepsilon) + \varepsilon \quad (\forall) t \in R$$

et réciproquement, si on a la deuxième relation, la première en résulte en remplaçant t par $t - \varepsilon$. Donc

$$F_1^*(t - \varepsilon) - \varepsilon \leq F_2^*(t), (\forall) t \in R \Leftrightarrow F_1^*(t) \leq F_2^*(t + \varepsilon) + \varepsilon (\forall) t \in R$$

De même

$$F_2^*(t) \leq F_1^*(t + \varepsilon) + \varepsilon (\forall) t \in R \Leftrightarrow F_2^*(t - \varepsilon) - \varepsilon \leq F_1^*(t) (\forall) t \in R$$

Il en résulte que $\rho^*(F_1^*, F_2^*) = \rho^*(F_2^*, F_1^*)$ où $F_1^*, F_2^* \in \mathcal{F}^*$.

Soient $\rho_1 = \rho^*(F_1^*, F_2^*)$ et $\rho_2 = \rho^*(F_2^*, F_3^*)$. On a :

$$F_1^*(t - \rho_1) - \rho_1 \leq F_2^*(t) \leq F_1^*(t + \rho_1) + \rho_1; t \in R$$

$$F_2^*(t - \rho_2) - \rho_2 \leq F_3^*(t) \leq F_2^*(t + \rho_2) + \rho_2; t \in R$$

En remplaçant dans la partie gauche de la première relation t par $t - \rho_2$, il en résulte :

$$F_1^*(t - \rho_1 - \rho_2) - \rho_1 \leq F_2^*(t - \rho_2) \Rightarrow F_1^*(t - \rho_1 - \rho_2) - \rho_1 - \rho_2 \leq F_3^*(t)$$

De même, en remplaçant t par $t + \rho_2$ dans la partie droite de la première relation et en tenant compte de la deuxième, on obtient

$$F_3^*(t) \leq F_1^*(t + \rho_1 + \rho_2) + \rho_1 + \rho_2 \quad t \in R$$

Donc, comme pour tout $t \in (0, 1)$, on a

$$F_1^*(t - \rho_1 - \rho_2) - \rho_1 - \rho_2 \leq F_3^*(t) \leq F_1^*(t + \rho_1 + \rho_2) + \rho_1 + \rho_2$$

il en résulte

$$\rho^*(F_1^*, F_3^*) \leq \rho_1 + \rho_2 = \rho^*(F_1^*, F_2^*) + \rho^*(F_2^*, F_3^*)$$

Proposition 2

Si $F^* \in \mathcal{F}^*$ et

$$F(x) = \inf \{t : F^*(t) \geq x\} \quad x \in R \quad (1)$$

alors F est une fonction de répartition.

Démonstration

Tout d'abord nous remarquons que $0 \leq F(x) \leq 1$ pour tout $x \in R$
(2). De même, comme F^* est continue à droite on a

$$F^*(F(x)) \geq x \text{ pour tout } x \in R \quad (3)$$

De plus

$$F(F^*(\tau)) = \inf \{t : F^*(t) \geq F^*(\tau)\} \leq \tau, \quad \tau \in (0, 1) \quad (4)$$

Evidemment F est nondécroissante. Prouvons que F est continue à gauche.

Pour cela soit $x_0 \in R$ et $x_n \uparrow x_0$. Notons $l = \lim_{n \rightarrow \infty} F(x_n)$. D'après la relation (3) on a $F^*(F(x)) \geq x_n$ et si $n \rightarrow \infty$, il en résulte que

$$F^*(l) \geq x_0 \Rightarrow l \geq F(x_0)$$

D'autre part

$$x_n \leq x_0 \Rightarrow F(x_n) \leq F(x_0) \Rightarrow l \leq F(x_0)$$

et la continuité à gauche est ainsi démontrée.

Si $x_n \uparrow \infty$ et si $l = \lim_{n \rightarrow \infty} F(x_n)$, alors de (3) on déduit

$$F^*(l) = +\infty \Rightarrow l \geq 1$$

et de (2) il résulte que $l \leq 1$, donc $\lim_{x \rightarrow \infty} F(x) = 1$.

Si $x_n \downarrow -\infty$, alors pour n suffisamment grand on a $x_n < F^*(\varepsilon)$ ($0 < \varepsilon < 1$ arbitraire), et donc d'après les relations (2) et (4) on a

$$0 \leq F(x_n) \leq F(F^*(\varepsilon)) \leq \varepsilon$$

Il en résulte que $\lim_{x \rightarrow -\infty} F(x) = 0$.

Proposition 3

Si F est une fonction de répartition et

$$F^*(t) = \begin{cases} \sup \{x : F(x) \leq t\} & \text{si } \{x : F(x) \leq t\} \neq \emptyset \\ -\infty & \text{si } \{x : F(x) \leq t\} = \emptyset \end{cases} \quad (5)$$

alors $F^* \in \mathcal{F}^*$.

Démonstration

Evidemment, F^* est nondécroissante sur R et finie sur l'intervalle $(0,1)$. De même il est facile à montrer que

$$F(F^*(t)) \leq t \quad t \in (0,1)$$

$$F^*(t) < a \Leftrightarrow F(a) > t \quad t \in (0, 1)$$

Soit $t_0 \in (0, 1)$ et $t_n \downarrow t_0$. Si $l = \lim_{n \rightarrow \infty} F^*(t_n)$ alors

$$F(F^*(t_n)) \leq t_n \Rightarrow F(l) \leq t_0 \Rightarrow F^*(t_0) \geq l$$

D'autre part

$$t_n > t_0 \Rightarrow F^*(t_n) \geq F^*(t_0) \Rightarrow l \geq F^*(t_0)$$

Il en résulte que

$$\lim_{t \downarrow t_0} F^*(t) = F^*(t_0) \quad t_0 \in (0, 1)$$

Si $F^*(0) > -\infty$ on montre semblablement — en tenant compte de la remarque $F(F^*(0)) = 0$ — que $\lim_{t \downarrow 0} F^*(t) = F^*(0)$.

Si $F^*(0) = -\infty$ et $t_n \downarrow 0$, soit $a \in R$ arbitraire.

Pour n suffisamment grand on a

$$t_n < F(a) \Leftrightarrow F^*(t_n) < a$$

Il en résulte que

$$\lim_{t \downarrow 0} F^*(t) = -\infty = F^*(0)$$

Soit $\mathcal{F}$ l'espace des fonctions de répartition et ρ la distance Paul Lévy sur $\mathcal{F}$ c'est à dire pour toute F_1 et F_2 de $\mathcal{F}$ on a :

$$\rho(F_1, F_2) = \inf \{ \varepsilon : F_1(x - \varepsilon) - \varepsilon \leq F_2(x) \leq F_1(x + \varepsilon) + \varepsilon, x \in R \}$$

Proposition

Les espaces métriques $(\mathcal{F}, \rho)$ et $(\mathcal{F}^*, \rho^*)$ sont isométriques.

Démonstration

Soit $\varphi : \mathcal{F} \rightarrow \mathcal{F}^*$ telle que $\varphi(F) = F^*$ où F^* est définie par la relation (5)

Montrons que φ est une isométrie entre les deux espaces métriques.

Tout d'abord remarquons que φ est une bijection. En effet si ψ est l'application définie par $\psi : F^* \in \mathcal{F}^* \rightarrow F \in \mathcal{F}$ comme dans la relation (1), alors

$$\psi(\varphi(F)) = F \quad (\forall) F \in \mathcal{F}; \quad \varphi(\psi(F^*)) = F^* \quad (\forall) F^* \in \mathcal{F}^* \quad (6)$$

Si $F \in \mathcal{F}$ et $F^* = \varphi(F) \in \mathcal{F}^*$, soit $F_0 = \varphi(F^*)$.

On a

$$F_0(x) = \inf \{ t : F^*(t) \geq x \} \quad x \in R$$

D'autre part on a

$$F^*(t) \geq x \Leftrightarrow F(x) \leq t \quad t \in [0, 1] \quad x \in R$$

et donc

$$F_0(x) = \inf \{ t : F(x) \leq t \} = F(x) \quad x \in R$$

De la même manière on démontre la deuxième relation (6). Pour montrer que

$$\rho(F_1, F_2) = \rho^*(\varphi(F_1), \varphi(F_2)) \text{ pour tout } F_1 \text{ et } F_2 \text{ de } \mathcal{F}$$

il suffit de voir que

$$a) F_1(x - \varepsilon) - \varepsilon \leq F_2(x) (\forall) x \in R \Leftrightarrow F_2^*(t) \leq F_1^*(t + \varepsilon) + \varepsilon (\forall) t \in [0, 1]$$

$$b) F_2(x) \leq F_1(x + \varepsilon) + \varepsilon (\forall) x \in R \Leftrightarrow F_1^*(t - \varepsilon) - \varepsilon \leq F_2^*(t) (\forall) t \in [0, 1]$$

où $F_i^* = \varphi(F_i)$, $i = 1, 2$.

Supposons satisfaite la première des relations a). En remplaçant dans celle-ci x par $F_2^*(t)$, $t \in (0, 1)$, il en résulte :

$$F_1(F_2^*(t) - \varepsilon) \leq F_2(F_2^*(t)) + \varepsilon \leq t + \varepsilon \Rightarrow F_1^*(t + \varepsilon) \geq F_2^*(t) - \varepsilon$$

Réciproquement, en remplaçant t par $F_2(x)$, $x \in R$ dans la deuxième relation a) on obtient

$$F_1^*(F_2(x) + \varepsilon) + \varepsilon \geq F_2^*(F_2(x)) \geq x \Rightarrow F_1(x - \varepsilon) \leq F_2(x) + \varepsilon$$

Si la première relation (b) est satisfaite et si l'on pose dans celle-ci $x = F_1^*(t - \varepsilon) - \varepsilon$ ($\varepsilon < t < 1$) alors

$$F_2(F_1^*(t - \varepsilon) - \varepsilon) \leq F_1(F_1^*(t - \varepsilon)) + \varepsilon \leq t \Rightarrow F_2^*(t) \geq F_1^*(t - \varepsilon) - \varepsilon.$$

Si $0 < t < \varepsilon$ l'inégalité est satisfaite car dans le membre droit apparaît $-\infty$.

Réciproquement si dans la deuxième relation b), on pose $t = F_1(x + \varepsilon) + \varepsilon$, il en résulte :

$$F_2^*(F_1(x + \varepsilon) + \varepsilon) \geq F_1^*(F_1(x + \varepsilon)) - \varepsilon \geq x \Rightarrow F_2(x) \leq F_1(x + \varepsilon) + \varepsilon$$

Corrolaire

L'espace métrique $(\mathcal{F}^*, \rho^*)$ est complet.

Proposition 5

La suite d'applications $(F_n^*)_{n \geq 1}$ de $\mathcal{F}^*$ converge vers $F^* \in \mathcal{F}^*$ par rapport à la distance ρ^* si et seulement si $\lim_{n \rightarrow \infty} F_n(t) = F(t)$ quelque soit le point de continuité $t \in (0, 1)$ de la fonction F^* .

Démonstration

Si $\rho_n = \rho^*(F_n^*, F^*)$ et $\rho_n \rightarrow 0$, alors quand n tends vers l'infini ($n \rightarrow \infty$) dans la relation

$$F^*(t - \rho_n) - \rho_n \leq F_n^*(t) \leq F^*(t + \rho_n) + \rho_n$$

il résulte $\lim_{n \rightarrow \infty} F_n^*(t) = F^*(t)$, si t est un point de continuité pour F^* .

Réciproquement nous montrerons que s'il existe un ensemble D de points de $(0,1)$ partout dense dans $(0,1)$ telle que $\lim_{n \rightarrow \infty} F_n^*(t) = F^*(t)$ pour tout $t \in D$, alors $\rho_n \rightarrow 0$ ($n \rightarrow \infty$). Pour ceci soit $\varepsilon > 0$ et $t_1, \dots, t_k$ dans D , tels que

$$0 = t_0 < t_1 < \dots < t_k < t_{k+1} = 1$$

et $t_{i+1} - t_i < \varepsilon$, $i = 0, 1, \dots, k$.

Pour n suffisamment grand on a

$$F^*(t_i) - \varepsilon \leq F_n^*(t_i) \leq F^*(t_i) + \varepsilon \quad i = 1, 2, \dots, k$$

Si $t \in (0,1)$, resp. $t \in [t_s, t_{s+1}]$, alors si $s < k$, on a

$$F_n^*(t) \leq F_n^*(t_{s+1}) \leq F^*(t_{s+1}) + \varepsilon \leq F^*(t + \varepsilon) + \varepsilon$$

Si $t \in (t_k, 1)$ l'inégalité $F_n^*(t) \leq F^*(t + \varepsilon) + \varepsilon$ reste vraie, car dans le membre droit apparaît $+\infty$. De même si $s > 0$ on a

$$F_n^*(t) > F_n^*(t_s) > F^*(t_s) - \varepsilon > F^*(t - \varepsilon) - \varepsilon$$

Si $s = 0$, l'inégalité reste encore vraie car dans le membre droit apparaît $-\infty$.

Il en résulte que pour n suffisamment grand on a

$$F^*(t - \varepsilon) - \varepsilon \leq F_n^*(t) \leq F^*(t + \varepsilon) + \varepsilon$$

c'est à dire

$$\rho^*(F_n^*, F^*) \leq \varepsilon$$

Corrolaire

Si $F, F_n \in \mathcal{F}$ ($n \geq 1$) et $F^* = \varphi(F)$; $F_n^* = \varphi(F_n)$ alors on a $F_n(x) \rightarrow F(x)$ ($n \rightarrow \infty$) pour tout point x de continuité de F , si et seulement si $F_n^*(t) \rightarrow F^*(t)$ pour tout point de continuité $t \in (0,1)$ de F^* .

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Centre de Statistique Mathématique
Bucarest

SUR L'ALGÈBRE DE DÉFORMATION ASSOCIÉE À UNE VARIÉTÉ DE RIEMANN

LIVIU NICOLESCU

Dans cette note on définit et on étudie l'algèbre de déformation associée à une variété de Riemann.

On met puis en évidence les champs et les courbes caractéristiques de cette algèbre.

Introduction

Les variétés différentiables, les applications différentiables, les champs tensoriels et les connexions linéaires qui interviennent à la suite sont supposés de la classe C^∞ .

Soit ∇ et $\overline{\nabla}$ deux connexions linéaires sur M et soit $A = \overline{\nabla} - \nabla$ le champ tensoriel de déformation [2]. Si on définit le produit de deux champs de vecteurs X et Y par la formule $X \circ Y = A(X, Y)$, les propriétés de distributivité de ce produit à la somme des champs des vecteurs sont immédiatement vérifiées. L'algèbre définie par la formule :

$$X \circ Y = A(X, Y), (\forall) X, Y \in \mathcal{X}(M),$$

s'appelle algèbre de déformation et on note par $\mathcal{U}(M, \nabla, \overline{\nabla})$ [6], [7]. Un champ $X \in \mathcal{X}(M)$ s'appelle champ caractéristique de l'algèbre. $\mathcal{U}(M, \nabla, \overline{\nabla})$ s'il existe une fonction $f \in \mathcal{F}(M)$, ainsi que $X \circ X = fX$ [2].

1. L'algèbre de déformation associée à une variété de Riemann

1. Soit (M, g) un espace de Riemann à deux dimensions dont le tenseur de Ricci S est non dégénéré et soit ∇ , resp. $\overline{\nabla}$, la connexion associée à g , resp. S . L'algèbre $\mathcal{U}(M, \nabla, \overline{\nabla})$ s'appelle algèbre de déformation associée de l'espace (M, g) .

Pour tout $X, Y \in \mathcal{X}(M)$ nous avons les relations :

$$(1.1) \quad \nabla_X Y - \nabla_Y X = [X, Y] = \overline{\nabla}_X Y - \overline{\nabla}_Y X$$

$$(1.1') \quad \nabla_X g = \nabla_X S = 0$$

Les relations (1.1) nous montrent que l'algèbre $\mathcal{U}(M, \nabla, \overline{\nabla})$ est commutative.

En tenant compte de (1.1') nous avons :

$$(1.1'') \quad X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$(1.1''') \quad X(S(Y, Z)) = S(\overline{\nabla}_X Y, Z) + S(Y, \overline{\nabla}_X Z),$$

En tenant compte de (1.1''') et de la relation $S = Cg$, nous avons :

$$(1.1^{iv}) \quad X(C)g(Y, Z) + CX(g(Y, Z)) = Cg(\nabla_X Y, Z) + Cg(Y, \nabla_X Z)$$

En employant (1.1'') et (1.1) nous avons : (1.2)

$$(1.2) \quad g(A(Y, X), Z) + g(Y, A(X, Z)) = X(\ln|C|)g(Y, Z)$$

En employant (1.2) il en résulte :

$$(1.3) \quad X(\ln|C|)g(Y, Z) + Y(\ln|C|)g(X, Z) - Z(\ln|C|)g(X, Y) = \\ = 2g(A(X, Y), Z)$$

Soit K le champ tensoriel de la courbure de déformation [6] donné par la formule

$$(1.4) \quad 4K(X, Y)Z = A(X, A(Y, Z)) - A(Y, A(X, Z))$$

En tenant compte de (1.3) et de (1.4) nous avons :

$$(1.4') \quad X(\ln|C|)g(A(Y, Z), W) + W(\ln|C|)g(Y, A(X, Z)) + \\ + A(Y, Z)(\ln|C|)g(W, X) - W(\ln|C|)g(X, A(Y, Z)) - \\ - Y(\ln|C|)g(A(X, Z), W) - A(X, Z)(\ln|C|)g(W, Y) - \\ - 2g(W, K(X, Y)Z) = 0, \quad (A) \quad X, Y, Z, W \in \mathcal{X}(M)$$

En tenant compte de (1.4') et (1.1) nous avons :

Théorème 1

L'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$ est associative si et seulement si pour tout $X, Y, Z, W \in \mathcal{X}(M)$ nous avons la relation

$$(1.5) \quad W(\ln|C|)g(Y, A(X, Z)) - W(\ln|C|)g(X, A(Y, Z)) + \\ + A(Y, Z)(\ln|C|)g(X, W) - A(X, Z)(\ln|C|)g(Y, W) + \\ + - X(\ln|C|)g(A(Y, Z), W) - Y(\ln|C|)g(A(X, Z), W) = 0.$$

Supposons que les connexions ∇ et $\bar{\nabla}$ possèdent les mêmes géodésiques. Alors il existe une 1-forme ω sur M tel que

$$(1.6) \quad A(X, Y) = \omega(X)Y + \omega(Y)X, \quad (\forall) X, Y \in \mathcal{X}(M)$$

En particulier, pour $Y = X$, nous avons

$$(1.6') \quad A(X, X) = 2\omega(X)X, \quad (\forall) X \in \mathcal{X}(M)$$

Si dans (1.3) on fait $Y = X$, il en résulte :

$$(1.7) \quad 2X(\ln|C|)g(X, Z) - Z(\ln|C|)g(X, X) = \\ = 2g(A(X, X), Z), \quad (\forall) X, Z \in \mathcal{X}(M)$$

En tenant compte de (1.6') et de (1.7). nous obtenons

$$(1.8) \quad 2X(\ln|C|)g(X, Z) - Z(\ln|C|)g(X, X) = \\ = 4\omega(X)g(X, Z), \quad (\forall) X, Z \in \mathcal{X}(M)$$

En particulier, pour $Z = X$, nous avons

$$(1.8') \quad X(\ln|C|)g(X, X) = 4\omega(X)g(X, X), \quad (\forall) X \in \mathcal{X}(M)$$

De (1.8') nous avons

$$(1.9) \quad X_p(\ln|C|) = 4\omega_p(X_p), \quad (\forall)p \in M, \quad (\forall)X_p \in T_p(M) - \{0\},$$

où $T_p M$ est l'espace tangent en chaque point $p \in M$. En tenant compte de (1.8) et de (1.9), nous obtenons

$$(1.10) \quad \omega_p(Z_p)g_p(X_p, X_p) = \omega_p(X_p)g_p(X_p, Z_p), \\ (\forall)p \in M, \quad (\forall)X_p, Z_p \in T_p(M) - \{0\}$$

Il est clair que pour tout $p \in M$ et pour tout $Z_p \in T_p(M) - \{0\}$, il existe $X_p \in T_p(M) - \{0\}$ tel que $g_p(X_p, Z_p) = 0$.

Alors de (1.10) nous obtenons $\omega_p(Z_p) = 0, (\forall)p \in M, (\forall)Z_p \in T_p(M) - \{0\}$. On déduit $\omega_p = 0, (\forall)p \in M$ et en tenant compte de (1.9), nous obtenons $C = \text{constante}$.

Réciproquement, si $C = \text{constante}$, alors ∇ et $\bar{\nabla}$ possèdent les mêmes géodésiques.

Nous avons donc le théorème suivant :

Théorème 2

Soit (M, g) un espace de Riemann à deux dimensions dont le tenseur de Ricci S est non dégénéré et soit ∇ , resp. $\bar{\nabla}$, la connexion associée à g , resp. S . Les propriétés suivantes sont équivalentes :

- 1) (M, g) est un espace de Riemann à courbure constante (non nulle).
2. ∇ et $\bar{\nabla}$ possèdent les mêmes géodésiques.

Soit $(U, h) = (x^1, x^2, \dots, x^n)$ une carte locale et soit $g_{ij}, \left| \begin{smallmatrix} i \\ jk \end{smallmatrix} \right|$, resp. R_{jk}^i , les composantes de g , ∇ , resp. $\bar{\nabla}$. Alors la relation (1.3) devient :

$$(1.3') \quad g \cdot \left(R_{ij}^s - \left| \begin{smallmatrix} s \\ ij \end{smallmatrix} \right| \right) = g_{jk} b_i + g_{ik} b_j - g_{ij} b_k$$

où $b_i = \frac{1}{2} \frac{\partial \ln |O|}{\partial x^i}$. En employant (1.3') il en résulte :

$$(1.3'') \quad R_{jk}^i = \left| \begin{smallmatrix} i \\ jk \end{smallmatrix} \right| + \delta_j^i b_k + \delta_k^i b_j - g_{jk} b^i,$$

où $b^i = g^{ij} b_j$. En tenant compte de (1.3'') on obtient les champs caractéristiques X^i de l'algèbre $\mathcal{U}\left(U, \left| \begin{smallmatrix} i \\ jk \end{smallmatrix} \right| R_{jk}^i \right)$;

$$(1.11) \quad X^i b^j - X^j b^i = 0$$

Nous supposons que (M, g) est un espace de Rimeann à courbure nonconstante. Alors, en tenant compte de (1.11) on obtient :

$$X^i = \lambda b^i$$

où λ est une fonction de $x^1, x^2, \dots, x^n$. Il en résulte que les courbes caractéristiques [2] de l'algèbre $\mathcal{U}\left(U, \left| \begin{smallmatrix} i \\ jk \end{smallmatrix} \right| R_{jk}^i \right)$ sont les trajectoires de champ b^i .

2. L'algèbre de déformation associée à une surface

Soit M une surface dans l'espace euclidien à trois dimensions avec la propriété que la deuxième forme fondamentale B est nondégénéré. Soit g la première forme fondamentale. On note avec ∇ , resp. $\bar{\nabla}$, la connexion associé à g , resp. B . L'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$ s'appelle algèbre de déformation associée de la surface M .

Puisque pour tout $X, Y \in \mathcal{X}(M)$ nous avons :

$$(2.1) \quad \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

$$(2.2) \quad \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = 0,$$

il en résulte que l'algèbre de déformation $\mathcal{U}(M, \nabla, \bar{\nabla})$ est commutative.

Soit $A = \bar{\nabla} - \nabla$ le champ tensoriel de déformation. Pour tout $X, Y, Z \in \mathcal{X}(M)$ nous avons :

$$(2.3) \quad (\nabla_X B)(Y, Z) = X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

En tenant compte de l'identité :

$$(2.4) \quad X(B(Y, Z)) = B(\bar{\nabla}_X Y, Z) + B(Y, \bar{\nabla}_X Z),$$

on obtient :

$$(2.5) \quad (\nabla_X B)(Y, Z) = B(A(Y, X), Z) = B(Y, A(X, Z)),$$

$$(2.5') \quad (\nabla_Y B)(Z, X) = B(A(Y, Z), X) + B(Z, A(X, Y)),$$

$$(2.5'') \quad (\nabla_Z B)(X, Y) = B(A(X, Z), Y) + B(X, A(Y, Z)).$$

En tenant compte de (2.5), (2.5') et de (2.5'') nous avons :

$$(2.6) \quad \begin{aligned} 2B(A(X, Y), Z) &= (\nabla_X B)(Y, Z) + \\ &+ (\nabla_Y B)(X, Z) - (\nabla_Z B)(X, Y) \end{aligned}$$

En tenant compte de l'équation de Codazzi :

$$(2.7) \quad (\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z),$$

il en résulte la formule :

$$(2.8) \quad B(A(X, Y), Z) = \frac{1}{2}(\nabla_X B)(Y, Z).$$

Dans un voisinage de coordonnées la formule (2.8) devient :

$$(2.8') \quad B_{is} A_{jk}^s = \frac{1}{2} \nabla_i B_{jk}$$

où A_{jk}^i , resp. B_{ij} , sont les composantes de A , resp. B .

En tenant compte de (2.8') nous avons [1], [3] :

$$(2.8'') \quad A_{jk}^i = \frac{1}{2} B^{is} \nabla_s B_{jk}$$

où $B^{ij} B_{jk} = \delta_k^i$.

En employant (2.8'') nous obtenons l'équation différentielle des courbes caractéristiques [2] de l'algèbre $\mathcal{U}(M, \nabla, \overline{\nabla})$:

$$(2.9) \quad (B^{1s} \nabla_s B_{jk} \delta_r^2 - B^{2s} \nabla_s B_{jk} \delta_r^1) \frac{dx^j}{dt} \frac{dx^k}{dt} \frac{dx^r}{dt} = 0$$

Il en résulte que par chaque point passent trois courbes caractéristiques [4], [5]. Nous dirons que le point $p \in M$ est de la première, deuxième ou troisième espèce suivant que les directions caractéristiques dans p sont toutes distinctes, seulement deux distinctes ou toutes confondues.

Nous dirons que M est de la première, deuxième, resp. troisième espèce si chaque point $p \in M$ est de la première, deuxième, resp. troisième espèce.

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Faculty of Mathematics
University of Bucharest

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En omettant certains des termes de la somme (2.1), on obtient la somme (2.2).

$$(2.2) \quad \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

Il est facile de vérifier que la somme (2.2) est égale à la somme (2.1) pour $x=y=1$.

On peut aussi remarquer que la somme (2.2) est égale à la somme (2.1) pour $x=1, y=0$ et $x=0, y=1$.

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SOME PROBLEMS IN NONLINEAR DIFFERENTIAL EQUATIONS WITH INTEGRAL PERTURBATIONS

B. G. PACHPATTE

Summary

A modified form of the Alekseev variation of constants formula coupled with the integral inequality recently established by this author is used to relate the behavior of the solution of systems of the form $x' = f(t, x, \lambda)$, λ in R^m and the perturbed system $y' = f(t, y, \lambda(t)) + g(t, y, \int_{\tau}^t k(t, s, y) ds)$.

1. Introduction

Let m and n be positive integers, S_c be the closed ball of radius c in R^m , and R^+ denote the nonnegative real numbers. In this paper we are interested in the relation between the solutions of the unperturbed differential system

$$(N) \quad x' = f(t, x, \lambda), \quad x(\tau) = x_0$$

and the solutions of the perturbed system

$$(P) \quad y' = f(t, y, \lambda(t)) + g\left(t, y, \int_{\tau}^t k(t, s, y) ds\right), y(\tau) = x_0.$$

We assume f and the Jacobian matrices f_x, f_{λ} are continuous for (t, x, λ) in $R^+ \times R^n \times S_c$ and denote the solution of (N) by $x(t, \tau, x_0, \lambda)$ for x_0 in R^n , $\lambda \in S_c$, and appropriate nonnegative numbers τ, t . We assume for any x_0 in R^n , λ in S_c , and $\tau \geq 0$ the solution $x(t, \tau, x_0, \lambda)$ exists for $t \geq \tau$, and $f(t, 0, \lambda) = 0$ for all $t \geq 0$ and $\lambda \in S_c$. Throughout this paper we assume that $\lambda(t)$ is a continuously differential function for $t \geq 0$ into S_c and k and g are continuous R^n valued functions defined on $R^+ \times R^+ \times R^n$ and $R^+ \times R^n \times R^n$ satisfying appropriate conditions. Further we assume that the solution $y(t)$ of (P) exists for all $t \geq \tau \geq 0$.

The nonlinear variation of constants formula of Alekseev [1] has been used in a number of studies on stability and asymptotic behavior of perturbed nonlinear systems (see, [2], [6], [7], [8], [10]). However, the case where f depends on λ is considered in very few references in English literature although these results are of great importance both in theory and applications. The only papers known to this author are [3]—[5] dealing with bounds and periodic behavior of the solutions of perturbed systems of differential equations. In the present paper results on the boundedness, asymptotic behavior and the rate of growth of the solutions of (P) are given under considerably less stringent hypotheses than in [3], [4].

2. Preliminaries

Our results in the next section depend on the following lemmas. Lemma 1 is a modified form of the Alekseev variation of constants formula recently established by T. G. Proctor in [4], and Lemma 2 is an integral inequality recently established by B.G. Pachpatte in [9].

Lemma 1

If $0 \leq \tau \leq t$ and $\lambda(t)$ is a solution of (P), then $y(t)$ satisfies

$$y(t) = x(t, \tau, x_0, \lambda(\tau)) + \int_{\tau}^t \left[\frac{\partial x}{\partial x_0}(t, s, y(s), \lambda(s)) g\left(s, y(s), \int_{\tau}^s k(s, n, y(n)) dn\right) + \frac{\partial x}{\partial \lambda}(t, s, y(s), \lambda(s)) \lambda'(s) \right] ds.$$

Lemma 2

Let $u(t)$, $a(t)$, $b(t)$, and $c(t)$ be real-valued nonnegative continuous functions defined on R^+ , for which the inequality

$$u(t) \leq u_0 + \int_0^t a(s) u(s) ds + \int_0^t b(s) u(s) \left(u(s) + \int_0^s c(\tau) u(\tau) d\tau \right) ds,$$

holds for all $t \in R^+$, where u_0 is a positive constant. Then

$$u(t) \leq u_0 \exp \left(\int_0^t [a(s) + b(s) E(s)] ds \right),$$

for all $t \in R^+$, where

$$E(t) = \frac{u_0 \exp \left(\int_0^t [a(\tau) + c(\tau)] d\tau \right)}{1 - u_0 \int_0^t b(\tau) \exp \left(\int_0^{\tau} [a(n) + c(n)] dn \right) d\tau}, \quad t \in R^+.$$

In our subsequent discussion we require the following definitions in terms of the behaviour of solutions of (N). For similar definitions the reader is referred to [2], [8].

Definition 1

The solution $x=0$ of (N) is said to be *globally uniformly stable* if there exists a constant $M > 0$ such that $|x(t, \tau, x_0, \lambda(\tau))| \leq M |x_0|$, for all $t \geq \tau \geq 0$ and $|x_0| < \infty$.

Definition 2

The solution $x = 0$ of (N) is said to be *exponentially asymptotically stable* if there exist constants $M > 0$, $\alpha > 0$, such that $|x(t, \tau, x_0, \lambda(\tau))| \leq M|x_0|e^{\alpha(t-\tau)}$, for all $t \geq \tau \geq 0$ and $|x_0|$ sufficiently small.

Definition 3

The solution $x = 0$ of (N) is said to be *uniformly slowly growing* if, and only if, for every $\alpha > 0$ there exists a constant M , possibly depending on α , such that $|x(t, \tau, x_0, \lambda(\tau))| \leq M|x_0|e^{\alpha(t-\tau)}$, for all $t \geq \tau \geq 0$ and $|x_0| < \infty$.

3. *Main Results.* In this section we state and prove our main results on the behavior of solution of (P) under some suitable conditions on the functions involved in (P). Our first theorem deals with the global uniform stability of the solutions of (P).

Theorem 1

Let the solution $x = 0$ of (N) be globally uniformly stable. Suppose that for y in R^n

$$(1) \quad \left| \frac{\partial x}{\partial \lambda}(t, s, y(s), \lambda(s)) \right| \leq a(s)|y(s)|, \quad t \geq s \geq 0,$$

$$(2) \quad \left| \frac{\partial x}{\partial x_0}(t, s, y(s), \lambda(s)) \right| \leq b(s)|y(s)|, \quad t \geq s \geq 0,$$

and that the functions g and k in (P) satisfy

$$(3) \quad |g(t, y, u)| \leq L[|y| + |u|], \quad t \geq 0,$$

$$(4) \quad |k(t, s, y)| \leq c(s)|y|, \quad 0 \leq s \leq t < \infty,$$

for $y, u \in R^n$, where $L > 0$ is a constant, and $a(t)$, $b(t)$, $c(t)$ are real-valued nonnegative continuous functions defined on R^+ such that

$$(5) \quad \int_{\tau}^{\infty} [a(s)|\lambda'(s)| + Lb(s)E_0(s)]ds < \infty,$$

in which

$$E_0(t) = \frac{M|x_0|\exp\left(\int_{\tau}^t [a(n)|\lambda'(n)| + c(n)]dn\right)}{1 - M|x_0|\int_{\tau}^t Lb(n)\exp\left(\int_{\tau}^n [a(s)|\lambda'(s)| + c(s)]ds\right)dn} \quad t \in R^+$$

where $M > 0$, $x_0 \neq 0$ are constants and $x_0 \in R^n$. Then any solution $y(t)$ of (P) is globally uniformly stable.

Proof

It is known by lemma 1 that if $y(t)$ is a solution of (P) then $y(t)$ satisfies the integral equation

$$(7) \quad y(t) = x(t, \tau, x_0, \lambda(\tau)) + \int_{\tau}^t \left[\frac{\partial x}{\partial \lambda}(t, s, y(s), \lambda(s)) \lambda'(s) + \frac{\partial x}{\partial x_0}(t, s, y(s), \lambda(s)) g\left(s, y(s), \int_{\tau}^s k(s, n, y(n)) dn\right) \right] ds,$$

for all t for which $y(t)$ is in R^n . Using (7), (1)–(4) together with the global uniform stability of the null solution of (N), we obtain

$$|y(t)| \leq M|x_0| + \int_{\tau}^t a(s)|\lambda'(s)||y(s)|ds + \int_{\tau}^t Lb(s)|y(s)| \left[|y(s)| + \int_{\tau}^s c(n)|y(n)|dn \right] ds.$$

Now, an application of Lemma 2 yields

$$|y(t)| \leq M|x_0| \exp \left(\int_{\tau}^t [a(s)|\lambda'(s)| + Lb(s)E_0(s)] ds \right),$$

where $E_0(t)$ is as defined in (6). The above estimation in view of the assumption (5) implies

$$|y(t)| \leq M_1|x_0|, \quad t \geq \tau \geq 0.$$

where $M_1 > 0$ is a constant. This proves the global uniform stability of the solution $y(t)$ of (P).

Our next theorem deals with the exponential asymptotic stability of the solution of (P) under some suitable conditions on the functions involved in (P).

Theorem 2

Let the solution $x = 0$ of (N) be exponentially asymptotically stable. Suppose that for y in R^n

$$(8) \quad \left| \frac{\partial x}{\partial \lambda}(t, s, y(s), \lambda(s)) \right| \leq e^{-\alpha(t-s)} a(s)|y(s)|, \quad t \geq s \geq 0,$$

$$(9) \quad \left| \frac{\partial x}{\partial x_0}(t, s, y(s), \lambda(s)) \right| \leq e^{-\alpha(t-s)} b(s)|y(s)|, \quad t \geq s \geq 0,$$

and that the functions g and k in (P) satisfy

$$(10) \quad |g(t, y, u)| \leq L[|y| + |u|], \quad t \geq 0,$$

$$(11) \quad |k(t, s, y)| \leq e^{-\alpha(t-s)} c(s) |y(s)|, \quad 0 \leq s \leq t < \infty,$$

for $y, u \in R^n$, where α and L are positive constants, and $a(t)$, $b(t)$, $c(t)$ are real-valued nonnegative continuous functions defined on R^+ such that

$$(12) \quad \int_{\tau}^{\infty} [a(s) |\lambda'(s)| + Lb(s) e^{-\alpha s} E_1(s)] ds < \infty,$$

in which

$$\begin{aligned} E_1(t) &= \\ &= \frac{M |x_0| e^{-\alpha \tau} \exp \left(\int_{\tau}^t [a(n) |\lambda'(n)| + c(n)] dn \right)}{1 - M |x_0| e^{\alpha \tau} \int_{\tau}^t L b(n) e^{-\alpha n} \exp \left(\int_{\tau}^n [a(\rho) |\lambda'(\rho)| + c(\rho)] d\rho \right) dn} \quad t \in R^+, \end{aligned} \quad (13)$$

where $M > 0$, $x_0 \neq 0$ are constants and $x_0 \in R^n$. Then any solution $y(t)$ of (P) is exponentially asymptotically stable.

Proof

It is known by Lemma 1 that if $y(t)$ is a solution of (P) then $y(t)$ satisfies the integral equation (7) for all t for which $y(t)$ is in R^n . Using (7), (8)–(11) together with the exponential asymptotic stability of the null solution of (N), we obtain

$$\begin{aligned} |y(t)| &\leq M |x_0| e^{-\alpha(t-\tau)} + \int_{\tau}^t e^{-\alpha(t-s)} a(s) |y(s)| |\lambda'(s)| ds \\ &+ \int_{\tau}^t e^{-\alpha(t-s)} b(s) |y(s)| L \left[|y(s)| + \int_{\tau}^s e^{-\alpha(s-n)} c(n) |y(n)| dn \right] ds. \end{aligned}$$

The above inequality can be written as

$$\begin{aligned} |y(t)| e^{\alpha t} &\leq M |x_0| e^{\alpha \tau} + \int_{\tau}^t a(s) |\lambda'(s)| |y(s)| e^{\alpha s} ds \\ &+ \int_{\tau}^t L b(s) e^{-\alpha s} |y(s)| e^{\alpha s} \left[|y(s)| e^{\alpha s} + \int_{\tau}^s c(n) |y(n)| e^{\alpha n} dn \right] ds. \end{aligned}$$

Now, applying Lemma 2 with $u(t) = |y(t)| e^{\alpha t}$, then multiplying by $e^{-\alpha t}$, we obtain

$$|y(t)| \leq M |x_0| e^{-\alpha(t-\tau)} \exp \left(\int_{\tau}^t [a(s) |\lambda'(s)| + Lb(s) e^{-\alpha s} E_1(s)] ds \right),$$

where $B_1(t)$ is as defined in (13). The above estimation in view of the assumption (12) implies

$$|y(t)| \leq M_2 |x_0| e^{-\alpha(t-\tau)}, \quad t \geq \tau \geq 0,$$

where $M_2 > 0$ is a constant. This proves the exponential asymptotic stability of the solution of (P).

Our Theorem 3 below deals with the rate of growth of the solution of (P).

Theorem 3

Let the solution $x = 0$ of (N) be uniformly slowly growing. Suppose that for y in R^n

$$(14) \quad \left| \frac{\partial x}{\partial \lambda}(t, s, y(s), \lambda(s)) \right| \leq e^{\alpha(t-s)} a(s) |y(s)|, \quad t \geq s \geq 0,$$

$$(15) \quad \left| \frac{\partial x}{\partial x_0}(t, s, y(s), \lambda(s)) \right| \leq e^{\alpha(t-s)} b(s) |y(s)|, \quad t \geq s \geq 0,$$

and that the functions g and k in (P) satisfy

$$(16) \quad |g(t, y, u)| \leq L[|y| + |u|], \quad t \geq 0,$$

$$(17) \quad |k(t, s, y)| \leq e^{\alpha(t-s)} c(s) |y(s)| \quad 0 \leq s \leq t < \infty,$$

for $y, u \in R^n$, where α and L are positive constants, and $a(t), b(t), c(t)$ are real-valued nonnegative continuous functions defined on R^+ such that

$$(18) \quad \int_{\tau}^{\infty} [a(s) |\lambda'(s)| + Lb(s) e^{\alpha s} B_2(s)] ds < \infty,$$

in which

$$(19) \quad E_2(t) = \frac{M |x_0| e^{-\alpha \tau} \exp \left(\int_{\tau}^t [a(n) |\lambda'(n)| + c(n)] dn \right)}{1 - M |x_0| e^{-\alpha \tau} \int_{\tau}^t Lb(n) e^{\alpha n} \exp \left(\int_{\tau}^n [a(\rho) |\lambda'(\rho)| + c(\rho)] d\rho \right) dn} \quad t \in R^+,$$

where $M > 0$, $x_0 \neq 0$ are constants and $x_0 \in R^n$. Then any solution $y(t)$ of (P) is uniformly slowly growing.

The proof of this theorem follows by the similar arguments as in the proof of Theorem 2 with suitable modifications, and hence we omit the details.

In previous papers [2], [10] the authors have studied the behavior relationships between the solutions of (N) and (P) when the integral term in (P) is absent and $\lambda(t) = 0$. In [6]–[8] the present author has studied the relationships between the solutions of (N) and (P) when

$\lambda(t) = 0$. Recently, in [4], [5] the authors have obtained bounds for solutions of (P) when the integral term in (P) is absent. The type of the hypotheses imposed on the perturbation term in (P) are general enough as compared to those given in [2]–[5] and [10]. Concerning the hypotheses on $\frac{\partial x}{\partial x_0}(t, \tau, x_0, \lambda(\tau))$ and $\frac{\partial x}{\partial \lambda}(t, \tau, x_0, \lambda(\tau))$, the present hypotheses are much less restrictive than in [4], [5].

4. An Example

In this section, we give a simple example to illustrate our Theorem 2. Consider the differential equations

$$(20) \quad x' = -x - \lambda(t)x^{\frac{1}{2}}, \quad t \geq \tau \geq 0, \quad x(\tau) = x_0 \geq 1,$$

and

$$(21) \quad y' = -y - \lambda(t)y^{\frac{1}{2}} + g\left(t, y, \int_{\tau}^t k(t, s, y) ds\right), \quad t \geq \tau \geq 0, \quad y(\tau) = x_0,$$

Suppose that the functions g and k in (21) satisfy the hypotheses and (11) of Theorem 2 with $\alpha = 1$ and let $\lambda(t) = e^{-\frac{1}{2}t}$. The solution $x(t) = x(t, \tau, x_0, \lambda(\tau))$ of (20) is given by

$$(22) \quad x(t) = e^{-t} \left\{ [x_0 e^{\tau}]^{\frac{1}{2}} - \frac{1}{2}(t - \tau) \right\}, \quad t \geq \tau \geq 0.$$

We observe from (22) that

$$(23) \quad |x(t)| \leq |x_0| e^{-(t-\tau)}, \quad t \geq \tau \geq 0.$$

From (23) it is clear that the null solution of (20) is exponentially asymptotically stable. Here

$$(24) \quad \frac{\partial x}{\partial x_0}(t, \tau, x_0, \lambda(\tau)) = \left[x_0^{\frac{1}{2}} e^{-(t-\tau)} - \frac{1}{2}(t - \tau) e^{-(t-\frac{1}{2}\tau)} \right] x_0^{-\frac{1}{2}},$$

and

$$(25) \quad \frac{\partial x}{\partial \lambda}(t, \tau, x_0, \lambda(\tau)) = 2 e^{-\frac{1}{2}t} [x_0 e^{\tau}]^{\frac{1}{2}} - \frac{1}{2}(t - \tau) \left\{ 1 + [x_0 e^{\tau}]^{\frac{1}{2}} - \frac{1}{2}(t - \tau) \right\}.$$

We observe from (24) and (25) that

$$(26) \quad \left| \frac{\partial x}{\partial x_0}(t, \tau, x_0, \lambda(\tau)) \right| \leq |x_0| e^{-(t-\tau)}, \quad t \geq \tau \geq 0,$$

and

$$(27) \quad \left| \frac{\partial x}{\partial \lambda}(t, \tau, x_0, \lambda(\tau)) \right| \leq |x_0| b(\tau) e^{-(t-\tau)}, \quad t \geq \tau \geq 0,$$

where $b(\tau) = 2[1 + e^{\frac{1}{2}\tau}]$. From (26) and (27) we see that

$$(28) \quad \left| \frac{\partial x}{\partial x_0}(t, s, y(s), \lambda(s)) \right| \leq e^{-(t-s)} |y(s)|, \quad t \geq s \geq 0,$$

and

$$(29) \quad \left| \frac{\partial x}{\partial \lambda}(t, s, y(s), \lambda(s)) \right| \leq e^{-(t-s)} b(s) |y(s)|, \quad t \geq s \geq 0.$$

From (28) and (29) it is clear that the hypotheses (8) and (9) of Theorem 2 are satisfied with $a(t) = 1$ and $\alpha = 1$. Thus all the hypotheses of Theorem 2 are satisfied and therefore the conclusion of Theorem 2 is true.

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Department of Mathematics
Deogiri College
Aurangabad (Maharashtra) India

IRREDUCIBLE POLYNOMIALS

MARIUS RĂDULESCU
SORIN RĂDULESCU

The purpose of the present paper is to provide some new results about irreducible polynomials in one or several variables.

Chapter 0 contains the basic notations and a series of basic theorems which will be used in the next chapters.

Chapter 1 deals with a few results and important methods used in the study of irreducible polynomials of several variables over an entire ring.

Chapter 2 is devoted to various results concerning irreducible polynomials over the rational integers.

Chapter 3 proposes to the reader some open problems interesting for study.

Notations and terminology

CHAPTER 0

Throughout this paper the word "ring" shall mean a commutative ring with an identity element. If A is a domain ring we shall denote by $K(A)$ its quotient field and by $\text{char } A$ its characteristic. If k is a field and n is a natural number we shall denote by k^n the set $\{x^n : x \in k\}$. If a and b are elements belonging to a ring A , their greatest common divisor will be denoted by (a, b) if it exists.

By M we understand the cardinal of the set M .

We shall use the following notations: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$

In an entire ring A by a "prime element" we shall understand a non null prime element of A .

By φ we shall understand the wellknown Euler function. Let A be a ring and $P(X_1, X_2, \dots, X_n)$ a polynomial over the ring A . By $P_0(X_1, X_2, X_3, \dots, X_n, Y)$ we shall mean the homogenized of the polynomial $P(X_1, X_2, \dots, X_n)$, so if

$$P(X_1, X_2, \dots, X_n) = \sum_{|\alpha| \leq m} a_\alpha X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n} \quad a_\alpha \in A,$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$$

$m = \deg P(X_1, X_2, \dots, X_n)$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ then

$$P_0(X_1, X_2, \dots, X_n, Y) = \sum_{|\alpha| \leq m} a_\alpha X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n} Y^{m-|\alpha|}$$

If $a \in \mathbb{Z}$ we shall denote by "div a " the set of all divisors of a . In the remainder of the paper the terminology adopted is that from [1].

We shall assume that the reader is familiar with the following results (See [1]):

0.1. Theorem (Eisenstein's Criterion)

Let A be a factorial ring. Let K be its quotient field. Let $f(X) = a_n X^n + \dots + a_0$ be a polynomial of degree $n \geq 1$ in $A[X]$. Let p a prime of A , and assume:

$a_n \not\equiv 0 \pmod{p}$, $a_i \equiv 0 \pmod{p}$ for all $i < n$, $a_0 \not\equiv 0 \pmod{p^2}$
Then $f(X)$ is irreducible in $K[X]$

0.2. Theorem

Let A be a factorial-ring, $K = K(A)$, $f(X) \in A[X]$ and $\deg f(X) \geq 1$
Then: $f(X)$ is irreducible in $A[X] \Leftrightarrow f(X)$ is irreducible in $K[X]$ and has content one.

0.3. Theorem

A is a factorial ring $\Leftrightarrow A[X]$ is a factorial ring.

0.4. Theorem

Let k be a field, $P(X) = X^n - a$, $n \in \mathbb{N}$, $a \in k$ and assume that the following conditions hold:

1°. $a \notin k^p$ for all prime numbers p such that $p \mid n$

2°. $a \notin -4k^4$ if $4 \mid n$

Then $P(X)$ is irreducible in $k[X] \Leftrightarrow P(X)$ satisfies to the conditions 1° and 2°.

0.5. Theorem

Let ζ be a primitive n -th root of unity. Then:

$$[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(n)$$

0.6. Theorem (Reduction Criterion)

Let A, B be domains, and let $\sigma : A \rightarrow B$ be a homomorphism. Let K, L be the quotient fields of A and B respectively. Let $f(X) \in A[X]$ be such that $f^\sigma(X) \neq 0$ and $\deg f^\sigma(X) = \deg f(X)$. If $f^\sigma(X)$ is irreducible in $L[X]$ then $f(X)$ does not have a factorization $f(X) = g(X) h(X)$ with

$$g(X), h(X) \in A[X] \text{ and } \deg g(X), \deg h(X) \geq 1.$$

0.7. Proposition

Let A be a domain and $P(X) \in A[X]$. Then the following statements are true :

- 1°. $P(X)$ irreducible $\Leftrightarrow P(X + a)$ irreducible $(\forall) a \in A$.
- 2°. $P(X)$ irreducible $\Leftrightarrow P(-X)$ irreducible
- 3°. $P(X)$ irreducible $\Leftrightarrow \tilde{P}(X)$ irreducible where $\tilde{P}(X) = X^n P(X^{-1})$ and $n = \deg P(X)$

CHAPTER 1

1.1. Proposition

Let A be a factorial ring, $P(X) \in A[X]$, $K = K(A)$, $a \in A$, $a \neq 0$ and $(a, P(0)) = 1$. Then : $Q(X) = P(aX)$ is irreducible in $A[X] \Leftrightarrow P(X)$ irreducible in $A[X]$.

Proof :

The necessity is clear. For if $(a, P(0)) = 1$ then $Q(X)$ has content one. Suppose that $P(X)$ is irreducible in $A[X]$ and $Q(X)$ is reducible in $A[X]$. By theorem 0.2 $Q(X)$ is reducible in $K[X]$ so $P(X) = Q(a^{-1}X)$ is reducible which is impossible.

1.2. Proposition

Let A be a domain and $P(X_1, X_2, \dots, X_n) \in A[X_1, X_2, \dots, X_n]$ irreducible. Then $P_0(X_1, X_2, \dots, X_n, Y)$ is irreducible in $A[X_1, X_2, \dots, X_n, Y]$.

Proof :

Suppose that $P_0(X_1, X_2, \dots, X_n, Y) = Q_1(X_1, X_2, \dots, X_n, Y) \cdot Q_2(X_1, X_2, \dots, X_n, Y)$, $Q_i(X_1, \dots, X_n, Y) \in A[X_1, X_2, \dots, X_n, Y]$, $i = 1, 2$ and both have $\deg \geq 1$. But $P(X_1, X_2, \dots, X_n) = P_0(X_1, X_2, \dots, X_n, 1)$ so there exists $i \in \{1, 2\}$ such that $Q_i(X_1, X_2, \dots, X_n, 1) = a$ is an invertible element belonging to A . We have therefore the following equality :

$$(*) \quad Q_i(X_1, \dots, X_n, Y) = (Y - 1)f(X_1, X_2, \dots, X_n, Y) + a$$

where $f(X_1, X_2, \dots, X_n, Y) \in A[X_1, X_2, \dots, X_n, Y]$.

But the polynomial $Q_i(X_1, \dots, X_n, Y)$ is homogeneous and the polynomial $f(X_1, X_2, \dots, X_n, Y)$ may be written as a sum of homogeneous polynomials $f_j(X_1, \dots, X_n, Y) \in A[X_1, \dots, X_n, Y]$ $j = 1, 2, \dots, k$ and for each $r, s \in \mathbb{N}$, $1 \leq r < s \leq k$ we have :

$$\deg f_r(X_1, \dots, X_n, Y) < \deg f_s(X_1, \dots, X_n, Y)$$

If we substitute this in $(*)$ we can note that :

$$Y f_k(X_1, \dots, X_n, Y) = Q_i(X_1, \dots, X_n, Y) \text{ and } Y f_j(X_1, \dots, X_n, Y) = f_{j+1}(X_1, \dots, X_n, Y) \quad 1 \leq j < k \text{ and } f_1(X_1, \dots, X_n, Y) = a. \text{ Hence } f(X_1, \dots, X_n, Y) = a(Y^{k-1} + \dots + 1) \text{ and } Q_i(X_1, \dots, X_n, Y) = aY^k.$$

Since Y is not a divisor of $P_o(X_1, \dots, X_n, Y)$ we obtain that $k = 0$ which is absurd. Therefore $P_o(X_1, \dots, X_n, Y)$ is irreducible in $A[X_1, X_2, \dots, X_n, Y]$

1.3. Proposition

Let A be a domain, $Q(X, Y) \in A[X, Y]$ and $P(Y, X) = X^n + Q(X, Y)$ and $n > \deg_x Q(X, Y)$. If there exists $a \in A$ such that the polynomial $P(X, a)$ is irreducible in $A[X]$ then $P(X, Y)$ is irreducible in $A[X, Y]$.

Proof:

Suppose that $R(X, Y)$ and $S(X, Y)$ are polynomials such that $P(X, Y) = R(X, Y) \cdot S(X, Y)$, $n > \max(\deg_x R(X, Y), \deg_x S(X, Y))$ and $\deg_x R(X, Y) \geq 1$, $\deg_x S(X, Y) \geq 1$. This implies that $P(X, a) = R(X, a) \cdot S(X, a)$ is reducible which contradicts with the hypothesis.

The next theorem is a variant of 0.1.

1.4. Theorem

Let A be a domain p a prime element of A and $P(X) = a_0 X^n + a_1 X^{n-1} + \dots + a_n$ a polynomial over A which satisfies the following conditions: $a_0 \not\equiv 0 \pmod{p}$, $a_i \equiv 0 \pmod{p}$ $i = 1, 2, \dots, n$, $a_n \not\equiv 0 \pmod{p^2}$ and each element belonging to A which is a divisor of a_i for all $i = 0, 1, 2, \dots, n$ is invertible.

Then $P(X)$ is irreducible in $A[X]$.

Proof:

Since p is a prime element of A then pA is a prime ideal. If $\sigma: A \rightarrow A/pA$ is the canonical homomorphism we have $P^\sigma(X) = \sigma(a_0)X^n$. If $P_i(X) \in A[X]$ $i = 1, 2$ such that $P(X) = P_1(X)P_2(X)$ then $P_1^\sigma(X) \cdot P_2^\sigma(X) = \sigma(a_0)X^n$ so $P_1^\sigma(X) = \alpha_1 X^r$ and $P_2(X) = \alpha_2 X^{n-r}$, $\alpha_1 \cdot \alpha_2 = \sigma(a_0) \neq 0$. If we suppose that $P(X)$ is reducible then r is distinct from 0 and n .

Hence p divide the constant terms of $P_1(X)$ and $P_2(X)$ and therefore $a_n \equiv 0 \pmod{p^2}$ which contradicts with the hypothesis. Therefore $P(X)$ is irreducible in $A[X]$.

1.5. Corollary

Let A be a factorial ring $K = K(A)$, $p \in A$ prime and $P(X) = a_0 X^n + a_1 X^{n-1} + \dots + a_n$ a polynomial over A which satisfies the following conditions: $a_0 \not\equiv 0 \pmod{p}$, $a_i \equiv 0 \pmod{p^i}$ $i = 1, 2, \dots, n-1$, $a_n \equiv 0 \pmod{p^{n-1}}$ and $a_n \not\equiv 0 \pmod{p^n}$. Then $P(X)$ is irreducible in $K[X]$.

Proof:

Let $a'_i \in A$ $i = 1, 2, \dots, n$ such that $a_i = p^i a'_i$ $i = 1, 2, \dots, n-1$, $a_n = p^{n-1} a'_n$, $a'_n \not\equiv 0 \pmod{p}$. Let $Q(X) = a'_n X^n + p a'_{n-1} X^{n-1} + \dots + p a'_1 X + p a_0$.

We note that $p \tilde{P}(X) = Q(pX)$ and that $Q(X)$ satisfies the conditions of the preceding theorem. Hence $Q(X)$ is irreducible in $A[X]$ and

therefore $Q(X)$ is irreducible in $K[X]$. Using the preceding equality and the fact that $Q(pX)$ is irreducible in $K[X]$ we obtain that $P(X)$ is irreducible in $K[X]$

1.6. Theorem

Let A be a domain, $p \in A$ prime and $P(X) = a_0 X^{n+1} + a_1 X^n + \dots + a_{n+1}$ a polynomial over A which satisfies the following conditions:

i) $P(a) \neq 0$ if $a \in K(A)$

ii) $a_0 \not\equiv 0 \pmod{p}$, $a_1 \not\equiv 0 \pmod{p}$, $a_i \equiv 0 \pmod{p}$

$i = 2, 3, \dots, n+1$ and $a_{n+1} \equiv 0 \pmod{p^2}$

iii) $a \in A$, $a \mid a_i$ $i = 0, 1, \dots, n+1$ implies a invertible,

Then $P(X)$ is irreducible in $A[X]$.

Proof:

We consider as in the proof of the theorem 1.4. the prime ideal pA . Using a similar argument as in the theorem 1.4. we obtain that $P^\sigma(X) = \sigma(a_0)X^{n+1} + \sigma(a_1)X^n = (\sigma(a_0)X + \sigma(a_1))X^n$.

If $P(X)$ is reducible then there exists $P_i(X) \in A[X]$, $\deg P_i(X) \geq 1$ $i = 1, 2$ such that $P(X) = P_1(X) \cdot P_2(X)$. Since $a_{n+1} \not\equiv 0 \pmod{p^2}$ then $\deg P_1(X) = 1$ or $\deg P_2(X) = 1$ which contradicts with the hypotheses because $P(a) \neq 0$ for each $a \in K(A)$.

1.7. Corollary

Let A be a factorial ring, $a_i \in A$ $i = 0, 1, 2, \dots, n+1$, $K = K(A)$, $p \in A$ prime, $P(X) = a_0 X^{n+1} + a_1 X^n + \dots + a_{n+1} \in A[X]$, $P(a) \neq 0$ if $a \in K$, $a_0 \not\equiv 0 \pmod{p}$, $a_1 \not\equiv 0 \pmod{p}$, $a_i \equiv 0 \pmod{p}$ $i = 2, 3, \dots, n+1$ and $a_{n+1} \not\equiv 0 \pmod{p^2}$. Then $P(X)$ is irreducible in $K[X]$.

1.8. Proposition

Let A be a factorial ring, $P(X) = X^n + naX + 1$, $a \in A$. If 2 is a prime element of A and $n = 2^k$, $k \geq 2$ then $P(X)$ is irreducible in $A[X]$.

Proof:

$P(X+1) = (X+1)^n + na(X+1) + 1 = X^n + C_n^1 X^{n-1} + \dots + C_n^{n-2} X^2 + n(a+1)X + na + 2$. We note that $2 \mid C_n^r$ if $r \in \{1, 2, \dots, n-1\}$ $2 \mid n(a+1)$, $2 \mid na + 2$; $4 \mid na + 2$. By the theorem 0.1. we obtain that $P(X+1)$ is irreducible. Therefore we conclude that $P(X)$ is irreducible in $A[X]$.

1.9. Corollary

Let A be a factorial ring, $a \in A$, 2 is a prime element of A , $n = 2^k$, $k \geq 2$ and $Q(Y) \in A[Y]$. Then $P(X, Y) = X^n + naQ(Y) + 1$ is irreducible in $A[X, Y]$.

1.10. Proposition

Let A be a factorial ring, p a prime element of $\mathbb{N}$ and $A, P(X) = X^p + paX + 1 \in A[X]$, $a \in A$ and $p \nmid a$.
If $p \geq 3$ then $P(X)$ is irreducible in $A[X]$.

Proof:

$$P(X-1) = (X-1)^p + pa(X-1) + 1 = X^p - C_p^1 X^{p-1} + \dots - C_p^{p-2} X^2 + p(a+1)X - pa.$$

We note that $p \mid C_p^r$ if $r \in \{1, 2, \dots, p-1\}$.

By the theorem 0.1. $P(X-2)$ is irreducible in $A[X]$ and we conclude that $P(X)$ is irreducible in $A[X]$.

1.11. Corollary

Let p a prime integer, $p \geq 3$ and $Q(Y) \in \mathbb{Z}[Y]$ with $p \nmid Q(Y)$. Then the polynomial $P(X, Y) = X^p + pXQ(Y) + 1$ is irreducible in $\mathbb{Z}[X, Y]$.

1.12. Theorem

Let A be a factorial ring, $m, n, r \in \mathbb{N}$ and $a, b \in A$ such that $a, b \neq 0$. If $\text{char } A \nmid m$ then $P(X, Y) = X^n Y^m + aX^n + bY^r$ is irreducible in $A[X, Y]$.

Proof:

The polynomial $P(X, Y) = X^n(Y^m + a) + bY^r \in (A[Y])[X]$ is irreducible iff the polynomial $bY^rX^n + (Y^m + a)$ is irreducible in $(A[Y])[X]$ (c.f. prop. 0.7). We note that $Y^m + a$ is square free i.e. it does not admit as factors square of polynomials which have $\deg \geq 1$. One can see that bY^r and $Y^m + a$ are coprime. By the theorems 0.1 and 0.3 we obtain that $P(X, Y)$ is irreducible in $A[X, Y]$.

1.13. Theorem

Let A be a domain, $K = K(A)$, $a, b \in A$, $ab \neq 0$ and assume that every element of A which is a divisor of a and b is inversable. If $m, n \in \mathbb{N}$ satisfies the conditions:

i) if p is prime and $p \mid (m, n)$ then $-ba^{-1} \notin K^p$

ii) if $4 \mid (m, n)$ then $ba^{-1} \notin 4K^4$

then the polynomial $P(X, Y) = aX^n + bY^m$ is irreducible in $A[X, Y]$

Proof:

Let $k = K(A[Y])$ and consider the polynomial $X^n + ba^{-1}Y^m \in k[X]$. We first prove that $-ba^{-1}Y^m \in k^p$ if p is prime such that $p \mid n$. If p is prime and $p \mid n$ and $-ba^{-1}Y^m \in k^p$ then $-ba^{-1}Y^m = [Q_1(Y) : Q_2(Y)]^p$ where $Q_i(Y) \in A[Y]$ $i=1,2$ are coprime. We get $Q_1(Y) = Y^q Q_3(Y)$ where $Q_3(Y) \in A[Y]$ and Y does not divide $Q_3(Y)$ and $q \in \mathbb{N}$. Therefore $-ba^{-1}Y^m [Q_2(Y)]^p = [Q_1(Y)]^p = Y^{pq} [Q_3(Y)]^p$. Since Y does not divide $Q_2(Y)$ then $pq = m$. Because $Q_2(Y)$ and $Q_3(Y)$ are coprime we get $Q_2(Y) = \beta \in A$

and $Q_3(Y) = \alpha \in A$. Therefore $Q_1(Y) = \alpha Y^q$. Let $\gamma = \beta \cdot \alpha^{-1} \in K$. Since $-ba^{-1} Y^m = (\gamma Y^q)^p = \gamma^p Y^{mq}$ then $\gamma^p = -ba^{-1} \in K^p$. Because $p \mid (m, n)$ and $-ba^{-1} \in K^p$ by (i), we get a contradiction.

Now we shall prove that $ba^{-1} Y^m \notin 4K^4$ if $4 \mid n$.

If this is not true by a similar argument as above we obtain: $ba^{-1} Y^m = 4[\alpha Q(Y)]^4$ where $\alpha \in K$ and $Q(Y) \in A[Y]$. Hence $4 \mid m$ and $(ba^{-1} - 4\alpha^4) Y^m = 0$ which implies that $ba^{-1} = 4\alpha^4 \in 4K^4$ whence we get a contradiction by (ii). By the theorem 0.4. we get $P(X, Y)$ is irreducible in $A[X, Y]$.

1.14. Corollary

Let A be a domain, $a, b \in A$, $a \neq 0 \neq b$ such that every common divisor of a and b is invertible. Then the polynomial $P(X, Y) = aX^n + bY^m$ with $m, n \in \mathbb{N}$, $(m, n) = 1$ is irreducible in $A[X, Y]$.

1.15. Corollary

Let A be a domain, $K = K(A)$ and $m, n \in \mathbb{N}$. Then: a) $(m, n) = 2$ and $-1 \notin K^2$ implies $X^m + Y^n$ is irreducible in $A[X, Y]$ b) $(m, n) = 2^r$, $r \geq 2$ and $-1 \notin K^2$, $1 \notin 4K^4$ implies $X^m + Y^n$ is irreducible in $A[X, Y]$.

c) If $\text{char } A = 2$ then: $X^m + Y^n$ irreducible in $A[X, Y] \Leftrightarrow (m, n) = 1$

1.16. Corollary

Let $m, n \in \mathbb{N}$. Then the following statements hold:

- a) $X^m + Y^n$ is irreducible in $\mathbb{Z}[X, Y] \Leftrightarrow (m, n) = 2^k$, $k \in \mathbb{N}_0$
- b) $X^m - Y^n$ is irreducible in $\mathbb{Z}[X, Y] \Leftrightarrow (m, n) = 1$.

1.17. Proposition

Let A be a domain, $m, n \in \mathbb{N}$, $\text{char } A \nmid n$ or $\text{char } A \nmid m$, $a, b \in A$, $a \neq 0 \neq b$. Then the polynomial $P(X, Y) = aX^n + bY^m$ does not admit squares of polynomials of $\deg \geq 1$ in $A[X, Y]$ as factors.

Proof:

We may assume, without loss of generality that $\text{char } A \nmid n$. We note that $\frac{\partial P}{\partial X}(X, Y) = n a X^{n-1} \neq 0$, whence the polynomials $P(X, Y)$

and $\frac{\partial P}{\partial X}(X, Y)$ have no common divisors and our proposition is proved.

1.18. Proposition

Let K be a field, $\text{char } K = p > 0$, $\alpha_i \in \mathbb{N}$, $p \mid \alpha_i$, $i = 1, 2, 3$, $a, b \in K$, $a \neq 0 \neq b$, $a \in K^p$ and $P(X, Y, Z) = X^{\alpha_1} - a Y^{\alpha_2} - b Z^{\alpha_3} \in K[X, Y, Z]$. Then the polynomial $P(X, Y, Z)$ is irreducible in $K[X, Y, Z]$.

Proof.

We shall prove that the polynomial $P(X, Y, Z) = X^{\alpha_1} - (aY^{\alpha_2} + bZ^{\alpha_3}) \in K(Y, Z)[X]$ satisfies the conditions of the theorem 0.4. Let $q \mid \alpha_1$, q prime and suppose that there exists $Q(Y, Z) \in K[Y, Z]$ such that we have

$$aY^{\alpha_2} + bZ^{\alpha_3} = [Q(Y, Z)]^q \quad (1)$$

If $q = p$ then $a, b \in K^p$ and we get a contradiction.

If $q \neq p$ then we consider the derivative of (1) respect to Z .

We have: $q[Q(Y, Z)]^{q-1} \cdot \frac{\partial Q}{\partial Z} = 0$ whence $\frac{\partial Q}{\partial Z}(Y, Z) = 0$.

Therefore $Q(Y, Z) = R(Y, Z^p)$ for some $R(Y, Z) \in K[Y, Z]$ whence $aY^{\alpha_2} + bZ^{\alpha_3} = [R(Y, Z^p)]^q$. If we denote $Z_1 = Z^p$ and $\alpha'_3 = \alpha_3 p^{-1}$ we have

$aY^{\alpha_2} + bZ_1^{\alpha'_3} = [R(Y, Z_1)]^q$. Applying repeatedly the same argument we may suppose that $aY^{\alpha_2} + bZ^{\alpha_3} = [Q(Y, Z)]^q$ where $\alpha_3 \not\equiv 0 \pmod{p}$. By the proposition 1.17 we obtain that $q = 1$, contradiction. Therefore it does not exist a polynomial $Q(Y, Z)$ such that (1) is satisfied. Let us assume that $4 \mid \alpha_1$ and there exists $Q(Y, Z) \in K[Y, Z]$ such that $-(aY^{\alpha_2} + bZ^{\alpha_3}) = 4[Q(Y, Z)]^4$.

Since $a \neq 0 \neq b$ then $p \neq 2$. Applying the preceding argument for $q = 2$ we get a contradiction. Therefore the conditions of the theorem 0.4. are satisfied for the polynomial $P(X, Y, Z)$ whence is irreducible in $K[X, Y, Z]$.

1.19. Theorem

Let K be a field, $n \geq 3$, $a_1, a_2, \dots, a_n \in K$ non-zero, $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{N}$ and $P(X_1, X_2, \dots, X_n) = a_1 X_1^{\alpha_1} + a_2 X_2^{\alpha_2} + \dots + a_n X_n^{\alpha_n} \in K[X_1, X_2, \dots, X_n]$ and $Q(X_1, X_2, \dots, X_n) = \sum_{r=1}^n a_r X_1^{\alpha_1} X_2^{\alpha_2} \dots X_{r-1}^{\alpha_{r-1}} X_{r+1}^{\alpha_{r+1}} \dots X_n^{\alpha_n} \in K[X_1, X_2, \dots, X_n]$.

Then the following statements are equivalent:

1. $P(X_1, X_2, \dots, X_n)$ is reducible in $K[X_1, X_2, \dots, X_n]$
2. $Q(X_1, X_2, \dots, X_n)$ is reducible in $K[X_1, X_2, \dots, X_n]$
3. $\text{char } K = p > 0$, $p \mid \alpha_i$ and there exists $\lambda \in K$, $\lambda \neq 0$ such that $\lambda \alpha_i \in K^p$ if $i = 1, 2, 3, \dots, n$.

Proof:

The fact that 1 is equivalent 2 is a consequence of the equality:

$P(X_1, X_2, \dots, X_n) = X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n} Q(X_1^{-1}, X_2^{-1}, \dots, X_n^{-1})$. Clear that statement 3 implies statement 1 and 2. We shall prove now that statement 1 implies statement 3. If $\text{char } K = 0$ then by the proposition 1.17 and the theorem 0.1 the polynomial $a_1 X_1^{\alpha_1} + a_2 X_2^{\alpha_2} + a_3 X_3^{\alpha_3}$ is irreducible in $K[X_1, X_2, X_3]$.

Applying repeatedly the theorem 0.1 we obtain that the polynomial $P(X_1, X_2, \dots, X_n)$ is irreducible in $K[X_1, X_2, \dots, X_n]$ but this fact contradicts with our assumption.

Therefore $\text{char } K = p > 0$. If there exists $i \in \{1, 2, \dots, n\}$ such that $p \nmid \alpha_i$ then applying as above the proposition 1.17. and the theorem 0.1. we obtain that the polynomial $P(X_1, X_2, \dots, X_n)$ is irreducible in $K[X_1, X_2, \dots, X_n]$ but this fact contradicts the statement 1.

Therefore $p \mid \alpha_i$ if $i = 1, 2, \dots, n$. Suppose now that there isn't any $\lambda \in K$, $\lambda \neq 0$ such that $\lambda a_i \in K^p$ if $i = 1, 2, \dots, n$. Hence $a_i a_1^{-1} \in K^p$ for some $i \in \{2, 3, \dots, n\}$. We may assume that $i = 2$. By the proposition 1.18. the polynomial $X_1^{\alpha_1} + a_2 a_1^{-1} \cdot X_2^{\alpha_2} + a_3 a_1^{-1} \cdot X_3^{\alpha_3}$ is irreducible in $K[X_1, X_2, X_3]$. Applying repeatedly the theorem 0.1 we obtain that the polynomial $P(X_1, X_2, \dots, X_n)$ is irreducible in $K[X_1, X_2, \dots, X_n]$ which contradict again with the statement 1. Therefore our theorem is proved.

We shall make the following notation:

$$S_k(X_1, X_2, \dots, X_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} X_{i_1} X_{i_2} \dots X_{i_k} \quad 1 \leq k \leq n$$

1.2.0. Theorem

Let K be a field, $n \geq 3$, $1 \leq r \leq n - 1$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{N}$. Then the following statements are equivalent:

1. The polynomial $S_r(X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_n^{\alpha_n})$ is reducible in $K[X_1, X_2, \dots, X_n]$.
2. $\text{char } K = p$ and $p \mid \alpha_i$ $i = 1, 2, \dots, n$.

Proof:

We note that the statement 2 implies the statement 1 is clear. We shall prove now that the statement 1 implies the statement 2. By the theorem 1.19 if the polynomial

$S_r(X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_n^{\alpha_n})$ is reducible in $K[X_1, X_2, \dots, X_n]$ for $r = 1$ or $r = n - 1$ then the statement 2 hold.

If $2 \leq r \leq n - 1$ then we have:

$$(*) \quad S_r(X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_n^{\alpha_n}) = X_1^{\alpha_1} \cdot S_{r-1}^{n-1}(X_2^{\alpha_2}, \dots, X_n^{\alpha_n}) + S_r^{n-1}(X_2^{\alpha_2}, \dots, X_n^{\alpha_n})$$

If the statement 2 is not true then by the theorem 1.19. the polynomial $S_r^{r-1}(X_1^{\alpha_1}, \dots, X_{r+1}^{\alpha_{r+1}})$ is irreducible in $K[X_1, X_2, \dots, X_{r+1}]$. If we put $n = r + 2$ in the equality (*) then by the theorem 0.1. the polynomial $S_r^{r+2}(X_1, X_2, \dots, X_n)$ is irreducible in $K[X_1, X_2, \dots, X_{r+2}]$. By complete induction we obtain that the polynomial $S_r(X_1^{\alpha_1}, \dots, X_n^{\alpha_n})$ is irre-

ducible in $K[X_1, X_2, \dots, X_n]$ but this contradicts with the statement 1. This proves that the statement 2 is true.

CHAPTER 2

2.1. Proposition

Let $P(X) \in \mathbb{Z}[X]$, $\deg P(X) = n$ and $a_1, a_2, \dots, a_{2n+1} \in \mathbb{Z}$ are distinct. If $P(a_i)$ is a prime number for all $i \in I = \{1, 2, \dots, 2n+1\}$ then the polynomial $P(X)$ is irreducible in $\mathbb{Z}[X]$.

Proof:

Suppose that $P(X) = Q(X) \cdot R(X)$ where $Q(X), R(X) \in \mathbb{Z}[X]$, $\deg Q(X) = q \geq 1$, $\deg R(X) = r \geq 1$, $q + r = n$. Let $I_1 = \{i \in I : Q(a_i) = 1\}$

$$I_2 = \{i \in I : Q(a_i) = -1\} \quad I_3 = \{i \in I : R(a_i) = 1\}$$

$$I_4 = \{i \in I : R(a_i) = -1\}.$$

We note that $I_1 \cup I_2 \cup I_3 \cup I_4 = I$ and $\bar{I}_k \leq q$ if $k = 1, 2$ and $\bar{I}_k \leq r$ if $k = 3, 4$. Since $2n + 1 = \sum_{k=1}^4 \bar{I}_k \leq 2(q + r) = 2n$ we get a contradiction.

2.2. Lemma

Let $P(X) \in \mathbb{Z}[X]$, $\deg P(X) = n$ and $a_1, a_2, \dots, a_{n+1} \in \mathbb{Z}$ are distinct such that $|P(a_i)| = 1$ if $i = 1, 2, \dots, n+1$. Then if $n \leq 3$ there exists such polynomials and if $n > 3$ there isn't.

Proof:

See [2] or [3]

2.3. Lemma

Let $Q(X) \in \mathbb{Z}[X]$, $\deg Q(X) \leq 3$, $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}$ are distinct such that $|Q(a_i)| = 1$ for all $i \in I = \{1, 2, 3, 4, 5\}$. Then $\deg Q(X) = 0$.

Proof:

Suppose that $\deg Q(X) \geq 1$. Let $I_1 = \{i \in I : Q(a_i) = 1\}$, $I_2 = \{i \in I : Q(a_i) = -1\}$. Since $I_1 \cup I_2 = I$ then $\bar{I}_1 = 3$ or $\bar{I}_2 = 3$. We may suppose that $\bar{I}_1 = 3$ and $I_1 = \{1, 2, 3\}$. Hence $Q(X) = a(X - a_4)(X - a_5)(X - a_3) + 1$ for some $a \in \mathbb{Z}$ and $Q(a_i) = -1$ if $i = 4, 5$. We notice that $|a| \neq 2$. Hence $|a| = 1$ and therefore $|(a_i - a_1)(a_i - a_2)(a_i - a_3)| = 2$ if $i = 4, 5$. We notice that the sets $\{a_1, a_2, a_3, a_4\}$ and $\{a_1, a_2, a_3, a_5\}$ consist of consecutive numbers and $\min(a_1, a_2, a_3) < a_i < \max(a_1, a_2, a_3)$ $i = 4, 5$ which led us to a contradiction. Therefore $\deg Q(X) = 0$.

2.4. Theorem

Let $P(X) \in \mathbb{Z}[X]$, $\deg P(X) = n - 5$ and $a_1, a_2, \dots, a_{2n-1} \in \mathbb{Z}$ are distinct. If $P(a_i)$ are prime for all $i \in I = \{1, 2, \dots, 2n - 1\}$ then the polynomial $P(X)$ is irreducible in $\mathbb{Z}[X]$.

Proof:

Suppose that $P(X) = Q(X) \cdot R(X)$ where $Q(X), R(X) \in \mathbb{Z}[X]$, $\deg Q(X) = q \geq 1$, $\deg R(X) = r \geq 1$. Let $I_1 = \{i \in I : |Q(a_i)| = 1\}$ and $I_2 = \{i \in I : |R(a_i)| = 1\}$. We notice that $I_1 \cup I_2 = I$ hence $\bar{I}_1 \geq n$ or $\bar{I}_2 \geq n$. Suppose that $\bar{I}_1 \geq n$. By the lemma 2.2 we obtain $1 \leq q \leq 3$. By the lemma 2.3 we get $q = 0$ which is a contradiction. This proves our theorem.

2.5. Proposition

Let $(a_i)_{i \in \mathbb{N}_0}$, $a_i \in \mathbb{Z} (\forall i \in \mathbb{N}_0)$, $a_0 \neq 0$ and

$$P_n(X) = a_0 X^{n+1} + a_1 C_{6n}^{3n} X^n + a_2 C_{6n+2}^{3n+1} X^{n-1} + \dots + a_n C_{8n-2}^{4n-1} X + X C_{8n}^{4n}.$$

Then there exists $N > C$ such that $n \in \mathbb{N}$, $n > N$ implies that the polynomial $P_n(X)$ is irreducible in $\mathbb{Q}[X]$.

Proof:

We shall use a well-known theorem of the theory of numbers which asserts that for each $n \in \mathbb{N}$, $n > N'$ (where N' is a sufficiently large natural number) there exists a prime number p_n such that $3n < p_n < 4n$. We notice that $C_{8n}^{4n} \not\equiv 0 \pmod{p_n^2}$ and $C_{6n+2k}^{3n+k} \not\equiv 0 \pmod{p_n}$ if $0 \leq k \leq n$. Let $N = \max(|a_0|, N')$. By the theorem 0.1 if $n > N$ then the polynomial $P_n(X)$ is irreducible in $\mathbb{Q}[X]$.

2.6. Theorem

Let $m, n \in \mathbb{N}$, $m = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$ (the factorization of m into prime factors) and $f_r(X) \in \mathbb{Z}[X]$ the r -th cyclotomic polynomial. Then $P(X) = f_m(X^n)$ is irreducible in $\mathbb{Z}[X]$ iff $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{N}_0$.

Proof:

By the theorem 0.5 we obtain that $\deg f_r(X) = \varphi(r)$

It is easy to verify that $f_{mn}(X) | f_m(X^n)$.

Then $P(X)$ is irreducible iff $\deg f_{mn}(X) = \deg f_m(X^n)$. Hence $\varphi(mn) = n\varphi(m)$.

We notice that this is true only if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{N}_0$.

2.7. Corollary,

$P(X) = X^n + 1$ is irreducible in $\mathbb{Z}[X]$ iff $n = 2^k$, $k \in \mathbb{N}_0$

2.8. Corollary

$P(X) = X^{2n} + X^n + 1$ is irreducible in $\mathbb{Z}[X]$ iff $n = 3^k$, $k \in \mathbb{N}_0$

2.9. Corollary

$P(X) = X^{2n} - X^n + 1$ is irreducible in $\mathbb{Z}[X]$ iff $n = 2^r \cdot 3^s$, $r, s \in \mathbb{N}_0$

2.10. Theorem

Let $P(X) \in \mathbb{Z}[X]$, $P(X) = X^n + a_1 X^{n-1} + \dots + a_n$ having the roots $x_k = \alpha_k + i\beta_k$, $\alpha_k, \beta_k \in \mathbb{R}$, $k \in \{1, 2, \dots, n\}$. Put $P_a(X) = P(X^2 - aX)$ and let $\prod_{k \in A} x_k \notin \text{div } a_n$ if A is a non empty set of $\{1, 2, \dots, n\}$.

Then $P(X)$ is irreducible in $\mathbb{Z}[X]$ iff there exists $N > 0$ such that if $a \in \mathbb{Z}$, $|a| \geq N$ implies $P_a(X)$ is irreducible in $\mathbb{Z}[X]$.

Proof:

The sufficiency is clear. For proving the necessity we shall compute the roots of the polynomial $P_a(X)$. We have:

$$x_{k,\pm} = \frac{1}{2} (a + \varepsilon (a^2 + 4x_k)^{\frac{1}{2}}) \text{ where } \varepsilon \in \{-1, 1\}$$

Let $\gamma_k, \delta_k \in \mathbb{R}$ such that $(a^2 + 4x_k)^{\frac{1}{2}} = \gamma_k + i\delta_k$. Hence we have $a^2 + 4\alpha_k = \gamma_k^2 - \delta_k^2$ and $4\beta_k = 2\gamma_k \cdot \delta_k$. If N is sufficiently large and $|a| \geq N$ then $\gamma_k \neq 0$ so we have: $\delta_k = 2\beta_k \gamma_k^{-1}$ and $a^2 + 4\alpha_k = \gamma_k^2 - (2\beta_k \gamma_k^{-1})^2$. Therefore γ_k satisfies the equation:

$$\gamma_k^4 - (a^2 + 4\alpha_k) \gamma_k^2 - 4\beta_k^2 = 0 \text{ and if } a^2 > 4 \max_{1 \leq k \leq n} |\alpha_k| \text{ then:}$$

$$\gamma_k = \left(\frac{1}{2} (((a^2 + 4\alpha_k)^3 + 16\beta_k^2)^{\frac{1}{2}} + a^2 + 4\alpha_k) \right)^{\frac{1}{2}} > 0$$

$$\delta_k = \left(\frac{1}{2} (((a^2 + 4\alpha_k)^3 + 16\beta_k^2)^{\frac{1}{2}} - a^2 - 4\alpha_k) \right)^{\frac{1}{2}} \cdot \text{sgn } \beta_k$$

If we make the following notation: $x_{k,+} = \frac{1}{2} (a + \gamma_k + i\delta_k)$ and $x_{k,-} =$

$= \frac{1}{2} (a - \gamma_k - i\delta_k)$ then we have: $x_{k,+} + x_{k,-} = a$ and $x_{k,+} \cdot x_{k,-} = -x_k$.

We notice that $\lim_{|a| \rightarrow \infty} \delta_k = 0$, $\lim_{|a| \rightarrow \infty} \gamma_k a^{-1} = 1$, $\lim_{|a| \rightarrow \infty} x_{k,+} \cdot a^{-1} = 1$ and $\lim_{|a| \rightarrow \infty} x_{k,-} \cdot a^{-1} = 0$.

The theorem will be proved if we show that there exists $N > 0$ such that $a \in \mathbb{Z}$, $|a| \geq N$ implies $\prod_{k \in B} x_{k,+} \cdot \prod_{k \in C} x_{k,-} \notin \text{div } a_n$ for any pair of sets B, C such that $\emptyset \neq B \cup C \subset \{1, 2, \dots, n\}$.

We shall prove more than that, precisely: the limit

$\lim_{|a| \rightarrow \infty} \prod_{k \in B} x_{k,+} \cdot \prod_{k \in C} x_{k,-}$ exists and does not belong to $\text{div } a_n$

The above limit is equal to: $\left(\prod_{k \in B \cap C} (-x_k) \right) \cdot \left(\lim_{|a| \rightarrow \infty} \prod_{k \in B-C} x_{k,+} \cdot \prod_{k \in C-B} x_{k,-} \right)$ which is equal to zero if $\overline{B-C} < \overline{C-B}$, to $\prod_{k \in C} (-x_k)$ if $\overline{B-C} = \overline{C-B}$ and to infinity if $\overline{B-C} > \overline{C-B}$.

2.11. Theorem

Let $P(X) \in \mathbb{Z}[X]$, $P(X) = X^n + a_1 X^{n-1} + \dots + a_n$ irreducible in $\mathbb{Z}[X]$, having the roots $x_k = \alpha_k + i\beta_k$, $\alpha_k, \beta_k \in \mathbb{R}$, $k \in I = \{1, 2, \dots, n\}$. Put $P_a(X) = P(X^2 - aX)$. If $P(X)$ satisfies the condition that there isn't a pair of distinct non empty sets $B, C \subset I$, $\overline{B} = \overline{C}$ such that $\sum_{k \in B} x_k = \sum_{k \in C} x_k$ then there exists $N > 0$ such that $a \in \mathbb{Z}$, $|a| \geq N$ implies that the polynomial $P_a(X)$ is irreducible in $\mathbb{Z}[X]$.

Proof:

By the theorem 2.10. we have to consider the case when there exists a non-empty subset A of I such that $\prod_{k \in A} x_k \in \text{div } a_n$.

Let $E(a) = \prod_{k \in B} x_{k,+} \cdot \prod_{k \in C} x_{k,-} = \prod_{k \in B \cap C} (-x_k) \cdot \prod_{k \in B-C} x_{k,+} \cdot \prod_{k \in C-B} x_{k,-}$.

Obviously we need study the case $\overline{B-C} = \overline{C-B}$.

$$\begin{aligned} E(a) &= \prod_{k \in C} (-x_k) \cdot \prod_{k \in B-C} (1 + \gamma_k a^{-1} + i\delta_k a^{-1}) \cdot \prod_{k \in C-B} (1 + \gamma_k a^{-1} + \\ &+ i\delta_k a^{-1})^{-1} = \prod_{k \in C} (-x_k) \cdot \prod_{k \in C-B} (1 + 2\alpha_k a^{-2} + 2i\beta_k a^{-2} + 0(a^{-4})) \cdot \\ &\cdot \prod_{k \in C} (1 - 2\alpha_k a^{-2} - 2i\beta_k a^{-2} + 0(a^{-4})) = \prod_{k \in C} (-x_k) \cdot \left(1 + 2a^{-2} \left(\sum_{k \in B-C} x_k - \right. \right. \\ &\quad \left. \left. - \sum_{k \in C-B} x_k \right) + 0(a^{-4}) \right). \end{aligned}$$

We used that: $\gamma_k a^{-1} = 1 + 2\alpha_k a^{-1} + 0(a^{-4})$ and $\delta_k = 2\beta_k a^{-1} + 0(a^{-3})$. We have $\lim_{|a| \rightarrow \infty} E(a) = \prod_{k \in C} (-x_k)$. We note that there exists $N > 0$ such that $a \in \mathbb{Z}$, $|a| \geq N$ implies $E(a) \neq \prod_{k \in C} (-x_k)$.

We conclude that the polynomial $P_a(X)$ is irreducible in $\mathbb{Z}[X]$ if $a \in \mathbb{Z}$, $|a| \geq N$.

2.12. Corollary

If p is a prime number and $P(X) = X^{p-1} + X^{p-2} + \dots + X + 1 \in \mathbb{Z}[X]$ then there exists $N \geq 0$ such that $a \in \mathbb{Z}$, $|a| \geq N$ implies that $P_a(X)$ is irreducible in $\mathbb{Z}[X]$.

2.13. Corollary

Let $P(X) = X^n + a_1 X^{n-1} + \dots + a_n \in \mathbb{Z}[X]$ an irreducible polynomial. If $n \in \{1, 2, 3, \dots\}$ then there exists $N > 0$ such that $a \in \mathbb{Z}$, $|a| \geq N$ implies $P_a(X)$ irreducible in $\mathbb{Z}[X]$.

CHAPTER 3

This chapter contains some questions and hypothesis concerning irreducible polynomials.

3.1. Problem

The polynomial $P(X) = [(X - a_1)(X - a_2) \dots (X - a_m)]^n + 1 \in \mathbb{Z}[X]$ ($a_i \in \mathbb{Z}$, $i = 1, 2, \dots, m$ are distinct) is irreducible iff $n = 2^k$, $k \in \mathbb{N}$? It is easy to verify that the polynomial $P(X)$ is irreducible in $\mathbb{Z}[X]$ if $n = 2$. The case $n = 4$ is more difficult.

Schur asserted and Brauer proved that in this case the polynomial $P(X)$ is irreducible.

3.2. Problem

For which values of $\alpha_1, \alpha_2, \dots, \alpha_6 \in \mathbb{N}$ the polynomial $P(X, Y, Z) = X^{\alpha_1} \cdot Y^{\alpha_2} + Y^{\alpha_3} \cdot Z^{\alpha_4} + Z^{\alpha_5} \cdot X^{\alpha_6}$ is irreducible in $\mathbb{Z}[X, Y, Z]$?

3.3. Problem

Let $P(X) \in \mathbb{Z}[X]$ irreducible and $P(X + iY) = P_1(X, Y) + iP_2(X, Y)$ where i is the square root in $\mathbb{C}$ of -1 , $P_1(X, Y), P_2(X, Y) \in \mathbb{Z}[X, Y]$. It is true that the polynomial $P_1(X, Y)$ is irreducible in $\mathbb{Z}[X, Y]$?

3.4. Problem

For which values of $n \in \mathbb{N}$ and $a \in \mathbb{Z}$, $a \neq 0$ the polynomial, $X^{pn} - X + a$ is irreducible in $\mathbb{Z}[X]$, where p is a prime number?

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Mathematical Faculty
University of Bucharest

2. Introduction

This paper is in continuation to my previous paper [3].

2.1. Simple relations :

In this article, we shall denote the left side of (1) by the symbol $S(x, y)$, and only those parameters in the definition of the function will be mentioned, which have been increased or decreased by an integer, e.g.

$$S = \left[\begin{array}{c|c|c} \begin{pmatrix} m_1, & 0 \\ p_1 - m_1, & q_1 \end{pmatrix} & a_1 + b, \dots, a_{p1}; & b_1, \dots, b_{q1} \\ \begin{pmatrix} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{pmatrix} & c_1, \dots, c_{p2}; & d_1, \dots, d_{q2} \\ \begin{pmatrix} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{pmatrix} & e_1, \dots, e_{p3}; & f_1, \dots, f_{q3} \end{array} \right] x, y$$

will be denoted by $S(a_1 + 1 | x, y)$.

From the definition of the generalised function of two variables (1). We obtain the following simple relations.

$$x \frac{\partial}{\partial x} S(x, y) + y \frac{\partial}{\partial y} (S(x, y) + a_j S(x, y)) = S(a_j + 1 | x, y), \quad (17)$$

where $j = 1, 2, \dots, m_1$.

$$x \frac{\partial}{\partial x} S(x, y) + y \frac{\partial}{\partial y} S(x, y) + a_j S(x, y) = -S(a_j + 1 | x, y), \quad (18)$$

where $j = m_1 + 1, \dots, p_1$.

$$x \frac{\partial}{\partial x} S(x, y) + y \frac{\partial}{\partial y} S(x, y) + (b_j - 1) S(x, y) = S(b_j - 1 | x, y) \quad (19)$$

where $j = 1, 2, \dots, q_1$.

$$x \frac{\partial}{\partial x} S(x, y) + (1 - c_j) S(x, y) = S(c_j - 1 | x, y), \quad (20)$$

where $j = 1, 2, \dots, m_2$.

$$-x \frac{\partial}{\partial x} S(x, y) + (e_j - 1) S(x, y) = S(e_j - 1 | x, y), \quad (21)$$

where $j = m_2 + 1, \dots, p_2$.

$$-x \frac{\partial}{\partial x} S(x, y) + d_j S(x, y) = S(d_j + 1 | x, y), \quad (22)$$

where $j = 1, 2, \dots, n_2$.

$$x \frac{\partial}{\partial x} S(x, y) - d_j S(x, y) = S(d_j + 1 | x, y), \quad (23)$$

where $j = n_2 + 1, \dots, q_2$.

$$y \frac{\partial}{\partial y} S(x, y) + (1 - e_j) S(x, y) = S(e_j - 1 | x, y), \quad (24)$$

where $j = 1, 2, \dots, m_4$.

$$-y \frac{\partial}{\partial y} S(x, y) + (e_j - 1) S(x, y) = S(e_j - 1 | x, y), \quad (25)$$

where $j = m_3 + 1, \dots, p_3$.

$$-y \frac{\partial}{\partial y} S(x, y) + f_j S(x, y) = S(f_j + 1 | x, y) \quad (26)$$

where $j = 1, 2, \dots, n_3$.

$$y \frac{\partial}{\partial y} S(x, y) - f_j S(x, y) = S(f_j + 1 | x, y), \quad (27)$$

where $j = n_3 + 1, \dots, q_3$.

2.3. Relations between functions contiguous to generalised function of two variables

In this article, we have obtained relations between contiguous functions and the generalised function of two variables, with the help of differential equations satisfied by the function $S(x, y)$. In my paper [2],

I have shown the function $S(x, y)$ satisfies the following differential equations,

$$\left[(-1)^{p_1+p_2-m_1-m_2-n_3} x \prod_{j=1}^{p_1} (\theta + \varphi + a_j - 1) \prod_{j=1}^{p_2} (\theta - c_j + 1) - \prod_{j=1}^{q_1} (b_j + \theta + \varphi - 1) \prod_{j=1}^{q_2} (\theta - d_j) \right] W = 0, \quad (28)$$

$$\left[(-1)^{p_1+p_2-m_1-m_2-n_3} y \prod_{j=1}^{p_1} (\theta + \varphi + a_j - 1) \prod_{j=1}^{p_2} (\varphi - e_j + 1) - \prod_{j=1}^{q_1} (b_j + \theta + \varphi - 1) \prod_{j=1}^{q_2} (\varphi - f_j) \right] W = 0, \quad (29)$$

where W stands for the right side of the equation (1), θ and φ denote $x \frac{\partial}{\partial x}$ and $y \frac{\partial}{\partial y}$ respectively.

Writing the equation (28) in the following form, we have

$$\begin{aligned} & \left[(-1)^{p_1+p_2-m_1-m_2-n_2} x \prod_{j=1}^{p_1} (\theta + \varphi + a_j - 1) \prod_{\substack{j=1 \\ j \neq h_1}}^{p_2} (\theta - c_j + 1) \right. \\ & \quad \left. (\theta + \varphi + a_{h_2}) \right] S(x, y) = \left[(-1)^{p_1+p_2-m_1-m_2-n_2} x \right. \\ & \quad \prod_{j=1}^{p_1} (\theta + \varphi + a_j - 1) \prod_{\substack{j=1 \\ j \neq h_1}}^{p_2} (\theta - c_j + 1) (c_{h_2} + c_{h_1} + \varphi - 1) + \\ & \quad \left. + \prod_{j=1}^{q_1} (b_j + \theta + \varphi - 1) \prod_{j=1}^{q_2} (\theta - d_j) \right] S(x, y), \quad (30) \end{aligned}$$

where $h_2 = 1, 2, \dots, m_1$.

On using the relation (19), (30) gives the following formula.

$$(-1)^{p_1+p_2-m_1-m_2-n_2} x \prod_{j=1}^{p_1} (\theta + \varphi + a_j - 1) \prod_{\substack{j=1 \\ j \neq h_1}}^{p_2} (\theta - c_j + 1)$$

$$S(a_{h_2} + 1 | x, y) = \left[(-1)^{p_1+p_2-m_1-m_2-n_2} x \prod_{j=1}^{p_1} (\theta + \varphi + a_j - 1) \right.$$

$$\left. \prod_{\substack{j=1 \\ j \neq h_1}}^{p_2} (\theta - c_j + 1) (a_{h_2} + c_{h_1} + \varphi - 1) + \prod_{j=1}^{q_1} (b_j + \theta + \varphi - 1) \prod_{j=1}^{q_2} (\theta - d_j) \right] S(x, y), \quad (31)$$

where $h_1 = 1, 2, \dots, p_2$ and $h_2 = 1, 2, \dots, m_1$.

In the similar manner, we obtain the following relations.

$$\begin{aligned}
 & (-1)^{p_1+p_2-m_1-m_2-n_2+1} x \prod_{j=1}^{p_1} (\theta + \varphi + a_j - 1) \prod_{\substack{j=1 \\ j \neq h_1}}^{p_1} (\theta - c_j + 1) \\
 & S(a_{h_2} + 1 | x, y) = \left[(-1)^{p_1+p_2-m_1-m_2-n_2} x \prod_{j=1}^{p_1} (\theta + \varphi + a_j - 1) \right. \\
 & \left. \prod_{\substack{j=1 \\ j \neq h_1}}^{p_2} (\theta - a_j + 1) (a_{h_2} + c_{h_1} + \varphi - 1) + \prod_{j=1}^{q_1} (b_j + \theta + \varphi - 1) \prod_{j=1}^{q_2} (\theta - d_j) \right] S(x, y),
 \end{aligned} \tag{32}$$

where $h_1 = 1, 2, \dots, p_2$; $h_2 = m_1 + 1, \dots, p_1$.

$$\begin{aligned}
 & (-1)^{p_1+p_2-m_1-m_2-n_2} x \prod_{\substack{j=1 \\ j \neq h_1}}^{p_1} (\theta + \varphi + a_j - 1) \prod_{j=1}^{p_2} (\theta - c_j + 1) \\
 & S(c_{h_2} - 1 | x, y) = \left[(-1)^{p_1+p_2-m_1-m_2-n_2} x \prod_{\substack{j=1 \\ j \neq h_1}}^{p_1} (\theta + \varphi + a_j - 1) \right. \\
 & \left. \prod_{j=1}^{p_2} (\theta - c_j + 1) (2 + a_{h_1} - c_{h_2} - \varphi) + \prod_{j=1}^{q_1} (\theta + \varphi + b_j - 1) \right. \\
 & \left. \prod_{j=1}^{q_2} (\theta - d_j) \right] S(x, y)
 \end{aligned} \tag{33}$$

where $h_1 = 1, 2, \dots, p_1$; $h_2 = 1, 2, \dots, m_2$.

$$\begin{aligned}
 & (-1)^{p_1+p_2-m_1-m_2-n_2+1} x \prod_{\substack{j=1 \\ j \neq h_1}}^{p_1} (\theta + \varphi + a_j - 1) \prod_{j=1}^{p_2} (\theta - c_j + 1) \\
 & S(c_{h_2} - 1 | x, y) = \left[(-1)^{p_1+p_2-m_1-m_2-n_2} x \prod_{\substack{j=1 \\ j \neq h_1}}^{p_1} (\theta + \varphi + a_j - 1) \right. \\
 & \left. \prod_{j=1}^{p_2} (\theta - c_j + 1) (2 + a_{h_1} - c_{h_2} - \varphi) + \prod_{j=1}^{q_1} (\theta + \varphi + b_j - 1) \prod_{j=1}^{q_2} (\theta - d_j) \right] S(x, y).
 \end{aligned} \tag{34}$$

where $h_1 = 1, 2, \dots, p_1$; $h_2 = m_2 + 1, \dots, p_2$.

Proceeding in the similar manner, one can obtain a number of other relations between the functions contiguous to the generalised function of two variables, with the help of the differential equations satisfied by the function.

2.3 Particular cases:

In this article, we have expressed several known functions of two variables in terms of the function $S(x, y)$. The importance of the function $S(x, y)$ is based largely on the possibility of expressing by means of the function $S(x, y)$ a number of known functions of two variables appearing in Mathematics. The formula developed for the function $S(x, y)$ becomes a key formula, from which a large number of relations can be deduced for Appelle's functions, Whittaker function of two variables etc. In our investigation, we shall make use of the result (1) and the following formula [2, p. 37].

$$S_1(x, y) = \sum_{u=1}^{m_2} \sum_{v=1}^{m_2} (x)^{c_u-1} (y)^{e_v-1} \frac{\prod_{j=1}^{m_2} \Gamma(c_u - c_j)}{\prod_{j=m_2+1}^{p_2} \Gamma(1 - c_u + c_j)} \times$$

$$\frac{\prod_{j=1}^{n_2} \Gamma(1 - c_u + d_j) \prod_{j=1, j \neq u}^{m_2} \Gamma(e_v - e_j) \prod_{j=1}^{n_2} \Gamma(1 - e_v + f_j)}{\prod_{j=n_2+1}^{q_2} \Gamma(c_u - d_j) \prod_{j=m_2+1}^{p_2} \Gamma(1 - e_v + e_j) \prod_{j=n_2+1}^{q_2} \Gamma(e_v - f_j)} \times$$

$$\times \frac{1}{\prod_{j=1}^{p_1} \Gamma(3 - a_j - e_v - c_u) \prod_{j=1}^{q_1} \Gamma(b_j + e_v + c_u - 2)} \times$$

$$F \left[\begin{array}{c|c} \begin{matrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 - 1 \\ p_3 - 1 \end{matrix} & \begin{matrix} 3 - b_1 - e_v - c_u, \dots, 3 - b_{q_1} - e_v - c_u \\ 1 + d_1 - c_u, \dots, 1 + d_{q_2} - c_u \\ 1 + f_1 - e_v, \dots, 1 + f_{q_3} - e_v \\ 3 - a_1 - e_v - c_u, \dots, 3 - a_{p_1} - e_v - c_u \\ 1 + c_1 - c_u, \dots, 1 + c_{p_2} - c_u \\ 1 + e_1 - e_v, \dots, 1 + e_{p_3} - e_v \end{matrix} \end{array} \middle| \begin{array}{c} \frac{(-1)^{q_1+q_2-m_2-n_2}}{x} \\ \frac{(-1)^{q_1+q_3-m_3-n_3}}{y} \end{array} \right] \quad (35)$$

The stars denote that the numbers $1 + c_u - c_u$ and $1 + f_v - f_v$ are to be omitted from the sequences $1 = c_1 - c_u, \dots, 1 + c_u - c_u, \dots, 1 + c_{p_3} - c_u$, and $1 + e_1 - e_v, \dots, 1 + e_v - e_v, \dots, 1 + e_{p_3} - e_v$ respectively. $S_1(x, y)$ stands for the function

$$S \left[\begin{matrix} \left[\begin{matrix} 0, & 0 \\ p_1, & q_1 \end{matrix} \right] \\ \left(\begin{matrix} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{matrix} \right) \\ \left(\begin{matrix} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{matrix} \right) \end{matrix} \middle| \begin{matrix} a_1, \dots, a_{p_1}; & b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; & d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; & f_1, \dots, f_{q_3} \end{matrix} \right] x, y \quad (36)$$

Taking $p_1 = m_1 = m$, $q_1 = n$, $m_2 = m_3 = p_2 = p_3 = q$, $n_2 = n_3 = 1$, $q_2 = q_3 = p + 1$, $d_1 = f_1 = 0$ and replacing c_j , d_j , e_j and f_j by $1 - c_j$, $1 - d_j$, $1 - e_j$ and $1 - f_j$ respectively in (1). Our generalised function two variables reduces to the hypergeometrics series of two variables defined by j. Kampé de Fériet [4, p. 100].

$$S \left[\begin{matrix} \left[\begin{matrix} m, & 0 \\ 0, & n \end{matrix} \right] \\ \left(\begin{matrix} q, & 1 \\ 0, & p \end{matrix} \right) \\ \left(\begin{matrix} q, & 1 \\ 0, & p \end{matrix} \right) \end{matrix} \middle| \begin{matrix} a_1, \dots, a_m; & b_1, \dots, b_n \\ 1 - c_1, \dots, 1 - c_q; & 0, 1 - d_1, \dots, 1 - d_p \\ 1 - e_1, \dots, 1 - e_q; & 0, 1 - f_1, \dots, 1 - f_p \end{matrix} \right] x, y =$$

$$= \frac{\prod_{j=1}^m \Gamma(a_j) \prod_{j=1}^q [\Gamma(c_j) \Gamma(e_j)]}{\prod_{j=1}^n \Gamma(b_j) \prod_{j=1}^p [\Gamma(d_j) \Gamma(f_j)]} \left[\begin{matrix} m \\ q \\ n \\ p \end{matrix} \middle| \begin{matrix} a_1, \dots, a_m \\ c_1, e_1, \dots, c_q, e_q \\ b_1, \dots, b_n \\ d_1, f_1, \dots, d_p, f_p \end{matrix} \right] - x, -y \quad (37)$$

Further, if we suppose that $q = p = 0$ in (37), we obtain

$$S \left[\begin{matrix} \left[\begin{matrix} m, & 0 \\ 0, & n \end{matrix} \right] \\ \left(\begin{matrix} 0, & 1 \\ 0, & 0 \end{matrix} \right) \\ \left(\begin{matrix} 0, & 1 \\ 0, & 0 \end{matrix} \right) \end{matrix} \middle| \begin{matrix} a_1, \dots, a_m; & b_1, \dots, b_n \\ ; & 0 \\ : & 0 \end{matrix} \right] x, y = \frac{\prod_{j=1}^m \Gamma(a_j)}{\prod_{j=1}^n \Gamma(b_j)}$$

$${}_mF_n \left[\begin{matrix} a_1, \dots, a_m; \\ b_1, \dots, b_n; \end{matrix} - (x+y) \right] \quad (38)$$

In (1), if we take $p_1 = q_1 = m_1 = 0$, then the function $S(x, y)$ reduces to a product of Maier's G -function [1, p. 207], i.e.

$$G_{p_1, q_1}^{n_1, m_1} \left(x \left| \begin{matrix} c_1, \dots, c_{p_1} \\ d_1, \dots, d_{q_1} \end{matrix} \right. \right) G_{p_2, q_2}^{n_2, m_2} \left(y \left| \begin{matrix} e_1, \dots, e_{p_2} \\ f_1, \dots, f_{q_2} \end{matrix} \right. \right) =$$

$$S \left[\begin{matrix} \left[\begin{matrix} 0, & 0 \\ 0, & 0 \end{matrix} \right] \\ \left(\begin{matrix} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{matrix} \right) \\ \left(\begin{matrix} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{matrix} \right) \end{matrix} \left| \begin{matrix} ; \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{matrix} \right. x, y \right] \quad (39)$$

By giving suitable values to the parameters in (37) and (39) we express known functions of two variables in terms of $S(x, y)$. We mention below only the results. The nomenclature of three functions is that given in Erdelyi [1, p. 224–227].

$$S \left[\begin{matrix} \left[\begin{matrix} 1, & 0 \\ 0, & 1 \end{matrix} \right] \\ \left(\begin{matrix} 1, & 0 \\ 0, & 0 \end{matrix} \right) \\ \left(\begin{matrix} 1, & 1 \\ 0, & 0 \end{matrix} \right) \end{matrix} \left| \begin{matrix} a & ; & b \\ 1-c; & 0 \\ 1-d; & 0 \end{matrix} \right. x, y \right] = \frac{\Gamma(a) \Gamma(c) \Gamma(d)}{\Gamma(b)} \times$$

$$F_1(a; c, d; b; -x, -y) \quad (40)$$

$$S \left[\begin{matrix} \left[\begin{matrix} 1, & 0 \\ 0, & 0 \end{matrix} \right] \\ \left(\begin{matrix} 1, & 1 \\ 0, & 1 \end{matrix} \right) \\ \left(\begin{matrix} 1, & 1 \\ 0, & 1 \end{matrix} \right) \end{matrix} \left| \begin{matrix} a & ; \\ 1-b; & 0, & 1-d \\ 1-e; & 0, & 1-f \end{matrix} \right. x, y \right] = \frac{\Gamma(a) \Gamma(b) \Gamma(e)}{\Gamma(d) \Gamma(f)} \times$$

$$F_2(a; b, e; d, f; -x, -y). \quad (41)$$

$$S \left[\begin{array}{c|c|c} \left[\begin{array}{cc} 1, & 0 \\ 0, & 0 \end{array} \right] & a & \\ \left(\begin{array}{cc} 1, & 2 \\ 0, & 0 \end{array} \right) & 1-b; & d, -d \\ \left(\begin{array}{cc} 1, & 2 \\ 0, & 0 \end{array} \right) & 1-c; & f, -f \end{array} \right] x, y = \sum_{d, -d} \sum_{f, -f} \Gamma(a+d+f) \\ \Gamma(b+d) \Gamma(c+f) \Gamma(-2d) \Gamma(-2f) x^d y^f \\ F_2(a+d+f; b+d, c+f; 1+2d, 1+2f; x, y). \quad (42)$$

$$S \left[\begin{array}{c|c|c} \left[\begin{array}{cc} 0, & 0 \\ 0, & 1 \end{array} \right] & & a \\ \left(\begin{array}{cc} 2, & 1 \\ 0, & 0 \end{array} \right) & 1-b, 1-c; & 0 \\ \left(\begin{array}{cc} 2, & 0 \\ 0, & 0 \end{array} \right) & 1-d, 1-f; & 0 \end{array} \right] x, y = \frac{\Gamma(b) \Gamma(c) \Gamma(d) \Gamma(f)}{\Gamma(a)} \times \\ \times F_3(b, d, c, f; a; -x, -y) \quad (43)$$

$$S \left[\begin{array}{c|c|c} \left[\begin{array}{cc} 2, & 0 \\ 0, & 0 \end{array} \right] & a, b; & \\ \left(\begin{array}{cc} 0, & 1 \\ 0, & 1 \end{array} \right) & & 0, 1-d \\ \left(\begin{array}{cc} 0, & 1 \\ 0, & 1 \end{array} \right) & & 0, 1-f \end{array} \right] x, y = \frac{\Gamma(a) \Gamma(b)}{\Gamma(d) \Gamma(f)} \times \\ \times F_4(a, b, d, f; -x, -y). \quad (44)$$

$$S \left[\begin{array}{c|c|c} \left[\begin{array}{cc} 2, & 0 \\ 0, & 0 \end{array} \right] & a, b; & \\ \left(\begin{array}{cc} 0, & 2 \\ 0, & 0 \end{array} \right) & & d, -d \\ \left(\begin{array}{cc} 0, & 2 \\ 0, & 0 \end{array} \right) & & f, -f \end{array} \right] x, y = \sum_{d, -d} \sum_{f, -f} \Gamma(a+d+f) \Gamma(b+d+f) \\ \Gamma(-2d) \Gamma(-2f) F_4(a, b; 1+2d, 1+2f; x, y). \quad (45)$$

$$S \left[\begin{array}{c} \left[\begin{array}{cc} 0, & 0 \\ 1, & 1 \end{array} \right] \\ \left(\begin{array}{cc} 1, & 1 \\ 0, & 0 \end{array} \right) \\ \left(\begin{array}{cc} 1, & 1 \\ 0, & 0 \end{array} \right) \end{array} \middle| \begin{array}{c} a; \quad b \\ c; \quad d \\ e; \quad f \end{array} \middle| x, \frac{1}{y} \right] = \frac{x^e y^{1-e} \Gamma(1-c+d) \Gamma(1-e+f)}{\Gamma(2-a-d-e) \Gamma(b+d+e-1)} \times$$

$$G_2(1-c+\frac{1}{2}d, \quad 1-e+f, \quad b+d+e-1, \quad 2-a-d-e; \quad -x, -y). \quad (46)$$

$$S \left[\begin{array}{c} \left[\begin{array}{cc} 0, & 0 \\ 1, & 0 \end{array} \right] \\ \left(\begin{array}{cc} 1, & 1 \\ 0, & 1 \end{array} \right) \\ \left(\begin{array}{cc} 1, & 0 \\ 0, & 2 \end{array} \right) \end{array} \middle| \begin{array}{c} a; \\ c; \quad d, \quad f \\ e; \quad g, \quad k \end{array} \middle| x, \frac{1}{y} \right] = x^d y^{1-e}$$

$$\frac{\Gamma(1-c+d)}{\Gamma(2-a-d-e) \Gamma(1-f+d) \Gamma(e-g) \Gamma(e-k)} \times$$

$$\times H_2(-1+a+d+e, \quad 1-c+d, \quad 1-e+g, \quad 1-e+k; \quad 1-f+d; \quad x, y). \quad (47)$$

$$S \left[\begin{array}{c} \left[\begin{array}{cc} 0, & 0 \\ 1, & 0 \end{array} \right] \\ \left(\begin{array}{cc} 0, & 1 \\ 0, & 1 \end{array} \right) \\ \left(\begin{array}{cc} 0, & 1 \\ 0, & 1 \end{array} \right) \end{array} \middle| \begin{array}{c} 1-k; \\ ; \quad m, \quad -m \\ ; \quad n, \quad -n \end{array} \middle| x, y \right] = \frac{1}{\Gamma(k-m-n) \Gamma(1+2m) \Gamma(1+2n)} \times$$

$$\times (xy)^{-\frac{1}{2}} e^{\frac{1}{2}(x+y)} M_{k,m,n}(x,y), \quad (48)$$

where $M_{k,m,n}(x,y)$ is a Whittaker function of two variables, according to Kampé de Fériet [4, p. 150] is

$$M_{k,m,n}(x,y) = x^{m+\frac{1}{2}} y^{n+\frac{1}{2}} e^{-\frac{1}{2}(x+y)}$$

$$\Psi_2(m+n-k+1; \quad 2m+1, \quad 2n+1; \quad x, y), \quad (49)$$

$$S \left[\begin{array}{c|c} \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix} & 1; \\ \begin{bmatrix} 0, & 1 \\ 0, & 1 \end{bmatrix} & ; \frac{1}{2}v, -\frac{1}{2}v \end{array} \middle| x, y \right] = e^{-(x+y)} I_v(2\sqrt{xy}). \quad (50)$$

$$S \left[\begin{array}{c|c} \begin{bmatrix} 0, & 0 \\ 1, & 0 \end{bmatrix} & 1; \\ \begin{bmatrix} 0, & 1 \\ 0, & 1 \end{bmatrix} & ; \frac{1}{2}v, -\frac{1}{2}v \end{array} \middle| x, y \right] = \frac{e^{x+y} I_v(2\sqrt{xy})}{\Gamma(1+v)\Gamma(-v)} \quad (51)$$

$$S \left[\begin{array}{c|c} \begin{bmatrix} 2, & 0 \\ 0, & 0 \end{bmatrix} & \frac{1}{2}, 1; \\ \begin{bmatrix} 0, & 1 \\ 0, & 1 \end{bmatrix} & ; \frac{1}{4} + \frac{1}{2}v, -\frac{1}{4} - \frac{1}{2}v \end{array} \middle| \frac{a^2}{x^2}, \frac{b^2}{x^2} \right] = \frac{x}{\sqrt{\pi ab}} Q_v \left(\frac{x^2 + a^2 + b^2}{2ab} \right). \quad (52)$$

$$S \left[\begin{array}{c|c} \begin{bmatrix} 1, & 0 \\ 1, & 0 \end{bmatrix} & 1+u, 1+v+u; \\ \begin{bmatrix} 0, & 1 \\ 0, & 3 \end{bmatrix} & ; 0, -v, -u, -u, -v \end{array} \middle| \frac{(x-y)^2}{x}, \frac{y}{x} \right] = -\frac{\Gamma(1+u)}{\pi\Gamma(1+v)} \sin(u+v)\pi(x)^{-\frac{u}{2}}(y)^{-\frac{v}{2}} \left(1 - \frac{y}{x}\right)^{-1-u-v} I_u(2\sqrt{x}) I_v(2\sqrt{y}). \quad (53)$$

$$S \left[\begin{array}{c} \left[\begin{array}{cc} 2, & 0 \\ 0, & 0 \end{array} \right] \\ \left(\begin{array}{cc} 0, & 1 \\ 0, & 3 \end{array} \right) \\ \left(\begin{array}{cc} 0, & 1 \\ 0, & 1 \end{array} \right) \end{array} \middle| \begin{array}{c} 1 + v, 1 + u + v; \\ \\ ; 0, -v, -u, -v - u \\ \\ ; 0, -v \end{array} \middle| \begin{array}{c} \\ \\ \frac{(x+y)^2}{x}, \frac{y}{x} \\ \\ \end{array} \right] = \\ = (x)^{-\frac{1}{2}u} (y)^{-\frac{1}{2}v} \left(1 + \frac{y}{x} \right)^{-1-v-u} J_u(2\sqrt{x}) I_v(2\sqrt{y}). \quad (54)$$

2.5 Recurrence relations :

In this article, we have obtained a few recurrence relations for the function $S(x, y)$. By giving suitable values to the parameters, we can get the recurrence relations for Appell's function and Whittaker functions. From (17) and (18), we get

$$(a_{h_1} - a_{h_2}) S(x, y) - S(a_{h_1} + 1 | x, y) - S(a_{h_2} + 1 | x, y) = 0, \quad (55)$$

where $h_1 = 1, 2, \dots, m_1$ and $h_2 = m_1 + 1, \dots, p_1$.

Now using (44), (55) gives

$$(b - a) F_1(a, b; d_1, d_2; x, y) + a F_4(a + 1, b; d_1, d_2; x, y) - b F_4(a, b + 1; d_1, d_2; x, y) = 0. \quad (56)$$

From (17) and (19), we get

$$(a_{h_1} - b_{h_2} + 1) S(x, y) + S(b_{h_2} - 1 | x, y) = S(a_{h_1} + 1 | x, y), \quad (57)$$

where $h_1 = 1, 2, \dots, m_1$; $h_2 = 1, 2, \dots, q_1$.

On using (40), (57) yields

$$(a - b + 1) F_1(a; d, e; b; x, y) - a F_1(a + 1; d, e; b; x, y) + b F_1(a; d, e; b - 1; x, y) \quad (58)$$

On using (18) and (19), we get

$$S(a_{h_1} + 1 | x, y) + S(b_{h_2} - 1 | x, y) + (a_{h_1} - b_{h_2} + 1) S(x, y) = 0, \quad (59)$$

where $h_1 = m_1 + 1, \dots, p_1$; $h_2 = 1, 2, \dots, q_1$.

Various other recurrence relations may be obtained by using the results (7) to (27).

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Department of Mathematics,
University of Ife,
Ile-Ife, Nigeria.

SUR UNE SOLUTION EN TENSIONS DE L'ÉLASTODYNAMIQUE LINÉAIRE

P. P. TEODORESCU

1°. Dans l'élastodynamique linéaire on a à résoudre un système de 15 équations avec 15 fonctions inconnues, respectivement : les équations de mouvement

$$(1) \quad \sigma_{ij,j} + F_i = \rho \ddot{u}_i,$$

les relations de Cauchy entre les déformations et les déplacements

$$(2) \quad \epsilon_{ij} = u_{(i,j)} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

et la loi de Hooke

$$(3) \quad \epsilon_{ij} = \frac{1}{2G} \left(\sigma_{ij} - \frac{\nu}{1 + \nu} \sigma_{kk} \delta_{ij} \right) = \frac{1}{E} [(1 + \nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij}].$$

Ci-dessus, σ_{ij} sont les tensions, ϵ_{ij} les déformations, u_i les déplacements, F_i les forces volumiques ($i, j = 1, 2, 3$), dans un système cartésien orthogonal de coordonnées $Ox_1x_2x_3$, ρ est la densité, δ_{ij} est le tenseur de Kronecker, E et G sont les modules d'élasticité longitudinale, respectivement transversale, ν est le coefficient de contraction transversale de Poisson et l'on a

$$(4) \quad G = \frac{E}{2(1 + \nu)}.$$

On utilise la convention de sommation d'Einstein, les indices à droite de la virgule signifient la dérivation par rapport aux variables correspondantes et les parenthèses mettent en évidence la partie symétrique du tenseur par rapport aux indices compris entre elles. Les points indiquent la dérivation par rapport au temps.

On considère le cas des corps homogènes, isotropes et linéairement élastiques soumis à des déformations infinitésimales.

Les problèmes de la théorie de l'élasticité (y compris les conditions initiales et les conditions sur le contour, qu'on ne va pas prendre en considération dans cet article) peuvent être résolus plus facilement en choisissant comme inconnues seulement les tensions ou seulement les déplacements et en éliminant toutes les autres inconnues entre les 15 équations mentionnées ci-dessus. On est conduit ainsi à la résolution de ces problè-

mes en tensions ou en déplacements. Dans ce qui va suivre nous allons nous rapporter seulement aux solutions en tensions de l'élastodynamique.

2°. En éliminant les déplacements et les déformations entre les équations (1)–(4), on obtient les équations du type de Beltrami-Michell (dérivées d'une manière indépendante par J. Ignaczak [1] et P. P. Teodorescu [3])

$$(5) \quad \square_2 \sigma_{ij} + \frac{1}{1 + \nu} \left[\partial_i \partial_j - \left(\frac{1}{c_1^2} - \frac{1}{2c_2^2} \right) \frac{\partial^2}{\partial t^2} \delta_{ij} \right] \sigma_{kk} = \\ = - 2F_{(i,j)} - \frac{\nu}{1 - \nu} F_{k,k} \delta_{ij}.$$

On introduit les opérateurs de d'Alembert

$$(6) \quad \square_k = \Delta - \frac{1}{c_k^2} \frac{\partial^2}{\partial t^2} \quad (k = 1, 2),$$

où Δ est l'opérateur de Laplace

$$(7) \quad \Delta = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}$$

et c_1, c_2 sont les vitesses de propagation longitudinale, respectivement transversale des ondes, données par

$$(8) \quad c_1^2 = \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} \frac{E}{\rho}, \quad c_2^2 = \frac{1}{2(1 + \nu)} \frac{E}{\rho};$$

on a la relation

$$(9) \quad (1 - 2\nu)c_1^2 = 2(1 - \nu)c_2^2$$

et la relation

$$(9') \quad 2(1 - \nu)\square_1 = \Delta + (1 - 2\nu)\square_2.$$

Le problème général de l'élastodynamique linéaire est ainsi réduit au système de 9 équations (1) et (5) pour les 9 fonctions $\sigma_{ij} = \sigma_{ij}(x_1, x_2, x_3; t)$ et $u_i = u_i(x_1, x_2, x_3; t)$; la loi de Hooke sous la forme

$$(10) \quad u_{(i,j)} = \frac{1}{E} [(1 + \nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij}]$$

doit être aussi ajoutée. La compatibilité du système ci-dessus est assurée.

Dans une première méthode de résoudre en tensions les problèmes de l'élastodynamique linéaire on part des équations de mouvement ; une représentation de la solution de ces équations doit vérifier aussi les équations du type de Beltrami-Michell.

Dans le cadre d'une seconde méthode, on cherche une représentation pour les équations du type de Beltrami-Michell ; cette représentation doit vérifier ensuite les équations de mouvement et la loi de Hooke.

Dans ce qui va suivre, on va considérer la seconde méthode de calcul.

3°. En partant des équations (5), dans l'absence des forces volumiques, on obtient la représentation

$$(11) \quad \sigma_{ij} = (2 + \nu) \square_1 F_{ij} - \left[\partial_i \partial_j - \left(\frac{1}{c_1^2} - \frac{1}{2c_2^2} \right) \frac{\partial^2}{\partial t^2} \delta_{ij} \right] F_{kk},$$

donnée par P. P. Teodorescu [5] pour l'état de tension ; les fonctions $F_{ij} = F_{ij}(x_1, x_2, x_3; t)$, de la classe C^4 , sont les composantes d'un tenseur potentiel symétrique du second ordre et doivent vérifier l'équation double des ondes

$$(12) \quad \square_1 \square_2 F_{ij} = 0.$$

La même représentation (11) peut être utilisée dans le cas des forces volumiques quelconques, mais les fonctions F_{ij} doivent vérifier l'équation

$$(13) \quad \square_1 \square_2 F_{ij} = - \frac{1}{2 + \nu} \left[2F_{(i,j)} + \frac{\nu}{1 - \nu} F_{kk} \delta_{ij} \right].$$

Mais cette représentation doit vérifier aussi l'équation de mouvement (1) et la loi de Hooke (10). C'est ainsi que l'équation (1) nous conduit à

$$(14) \quad \square_1 [(2 + \nu) F_{ij,j} - F_{kk,i}] - \frac{1}{c_2^2} \ddot{F}_{kk,i} = \rho \ddot{u}_i.$$

On peut écrire

$$(15) \quad u_i = - \frac{1}{2G} F_{kk,i} + \varphi_i$$

et

$$(16) \quad \square_1 [(2 + \nu) F_{ij,j} - F_{kk,i}] = \rho \ddot{\varphi}_i,$$

où les fonctions $\varphi_i = \varphi_i(x_1, x_2, x_3; t)$, de la classe C^2 , vérifient l'équation des ondes transversales

$$(17) \quad \square_2 \varphi_i = 0.$$

En observant que

$$(18) \quad \varepsilon_{ij} = -\frac{1}{2G} F_{kk,ij} + \varphi_{(i,j)}$$

et en tenant compte de la loi de Hooke (10) et de la représentation (11) on arrive à la condition

$$(16') \quad \square_1[(2 + \nu)F_{ij} - \nu F_{kk} \delta_{ij}] = 2G \varphi_{(i,j)}$$

Les relations (16) et (16') doivent être vérifiées par les fonctions F_{ij} , qui vérifient l'équation (12), et par les fonctions φ_i , qui vérifient l'équation (17).

De (16'), il résulte

$$(19) \quad G\varphi_{k,k} = (1 - \nu)\square_1 F_{kk};$$

tenant compte de cette dernière relation, de même que de l'équation (17), on observe que la condition (16) est une conséquence de la condition (17'). Ainsi que les fonctions φ_i , qui vérifient l'équation des ondes transversales, sont déterminées par les relations (16') et la représentation donnée ci-dessus est complète pour le système fondamental d'équations de l'élastodynamique, dans l'absence des forces volumiques.

En substituant

$$(20) \quad F_{ij} = \frac{1}{2 + \nu} \left[\chi_{ij} + \frac{\nu}{2(1 - \nu)} \chi_{kk} \delta_{ij} \right],$$

où les fonctions $\chi_{ij} = \chi_{ij}(x_1, x_2, x_3; t)$, de la classe C^4 , sont les composantes d'un tenseur symétrique du second ordre, on obtient la représentation

$$(21) \quad \sigma_{ij} = \square_1 \chi_{ij} - \frac{1}{2(1 - \nu)} (\partial_i \partial_j - \nu \square_2 \delta_{ij} \chi_k),$$

$$(21') \quad u_i = -\frac{1}{4(1 - \nu)G} \chi_{kk,i} + \varphi_i,$$

où les fonctions de tension vérifient la double équation des ondes

$$(22) \quad \square_1 \square_2 \chi_{ij} = 0$$

et les fonctions φ_i , qui vérifient l'équation des ondes transversales, sont données par le système

$$(23) \quad 2G\varphi_{(i,j)} = \square_1 \chi_{ij}.$$

En utilisant la substitution

$$(24) \quad \Omega = - \frac{1}{2(1-\nu)} \chi_{kk},$$

où $\Omega = \Omega(x_1, x_2, x_3; t)$, et en introduisant les fonctions $\Theta_{ij} = \Theta_{ij}(x_1, x_2, x_3; t)$, solutions de l'équation des ondes transversales, par la relation

$$(24') \quad \Theta_{k(i,j)k} = - \frac{1}{2} \square_1 \chi_{ij},$$

on obtient une représentation du type de Schaefer [2]

$$(25) \quad \sigma_{ij} = -\nu \square_2 \Omega \delta_{ij} + \Omega_{,ij} - 2\Theta_{k(i,j)k},$$

$$(25') \quad u_i = \frac{1}{2G} (\Omega_{,i} - 2\Theta_{ij,j}),$$

donnée par P. P. Teodorescu [4], en partant des équations de mouvement.

4°. Les représentations données ci-dessus sont équivalentes, mais chacune est plus convenable pour un problème aux limites ou pour un autre. Ces représentations sont complètes pour un domaine simplement connexe; dans le cas d'un domaine multiplement connexe on doit ajouter des termes supplémentaires, correspondant à certaines singularités.

De même, on observe que les deux méthodes de résoudre en tensions les problèmes de l'élastodynamique linéaire sont équivalentes.

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Faculté de Mathématiques
Université de Bucharest

PROPERTIES OF A FUNCTIONAL DEFINED ON THE PATHS SPACE OF A RIEMANNIAN MANIFOLD

CONSTANTIN UDRIȘTE

Let Ω be the set of piecewise differentiable regular paths which join two points in a Riemannian manifold. Starting from [1]—[7] we shall show that the functional

$$E(\omega; p) = \int_0^1 \left\| \frac{d\omega}{dt} \right\|^p dt, \quad \omega \in \Omega, \quad p \in \mathbb{R} - \{0\},$$

has analogous properties with those of the length functional ($p = 1$) and of the energy functional ($p = 2$). Our results can be generalized as properties of positive homogeneous Lagrangians which do not contain the parameter t explicitly; two of possible Theorems are found in [7].

§ 1. Preliminaries

Let M be a differentiable manifold, x, y be two (distinct or not) points in M , Ω be the set of all piecewise differentiable regular paths $\omega: [0, 1] \rightarrow M$ which join $x = \omega(0)$ with $y = \omega(1)$ and $T_\omega \Omega$ be the vector space consisting of all piecewise differentiable vector fields W along ω for which $W(0) = W(1) = 0$. The set Ω is similar to a manifold and $T_\omega \Omega$ plays the role of the tangent space.

Let $\alpha: (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M$ be a variation of ω , $W(t) = \frac{\partial \alpha}{\partial u}(0, t)$ the variation vector field associated with α and $\bar{\alpha}(u) \in \Omega$ the path defined by $\bar{\alpha}(u)(t) = \alpha(u, t)$. Obviously $W = \frac{d\bar{\alpha}}{du}(0) \in T_\omega \Omega$. Let $F: \Omega \rightarrow \mathbb{R}$ be a functional with real values which satisfies those conditions guaranteeing the existence of $\frac{d}{du} F(\bar{\alpha}(u))|_{u=0}$, whatever $\bar{\alpha}$. The equality

$$F_*(W) = \frac{d}{du} F(\bar{\alpha}(u))|_{u=0} \stackrel{\text{notation}}{=} \frac{dF}{du}(0)$$

defines a linear functional $F_*: T_\omega \Omega \rightarrow \mathbb{R}$. If $\frac{d}{du} F(\bar{\alpha}(u))|_{u=0} = 0, \forall \bar{\alpha}$, then ω is called a *critical point* of F .

Let γ be a critical point of the functional F and $W_1, W_2 \in T_\gamma \Omega$. We choose a 2-parameter variation of γ , $\alpha: U \times [0, 1] \rightarrow M$, $(0, 0) \in U \subset R^2$, so that

$$\alpha(0, 0, t) = \gamma(t), \quad \frac{\partial \alpha}{\partial u_1}(0, 0, t) = W_1(t), \quad \frac{\partial \alpha}{\partial u_2}(0, 0, t) = W_2(t),$$

and we denote with $\bar{\alpha}(u_1, u_2) \in \Omega$ the path defined by $\bar{\alpha}(u_1, u_2)(t) = \alpha(u_1, u_2, t)$. The bilinear functional $F_{**}: T_\gamma \Omega \times T_\gamma \Omega \rightarrow R$ defined by

$$F_{**}(W_1, W_2) = \frac{\partial^2 F(\bar{\alpha}(u_1, u_2))}{\partial u_1 \partial u_2} \Big|_{(0, 0)} \stackrel{\text{notation}}{=} \frac{\partial^2 F}{\partial u_1 \partial u_2}(0, 0)$$

on condition that the middle term exists whatever $\bar{\alpha}$, is called the *Hessian* of F .

§2. The functional E and the first variation formula

Let M be a Riemannian manifold, $\langle V, W \rangle$ the scalar product of two vector fields determined by the Riemannian metric, $\|V\| = \sqrt{\langle V, V \rangle}$ the norm of V , $L(\omega) = \int_0^1 \left\| \frac{d\omega}{dt} \right\| dt$ the length of the path ω which unites the points x, y , and $d(x, y) = \inf_{\omega \in \Omega} L(\omega)$ the distance from x to y . We define the functional

$$E(\omega; p) = \int_0^1 \left\| \frac{d\omega}{dt} \right\|^p dt, \quad \omega \in \Omega, \quad p \in R - \{0\}.$$

Particularly $E(\omega; 1)$ is the length of ω and $E(\omega; 2)$ is called the energy of ω [4, §12–§16]. Some modifications in arguments of [4] permit us to prove that $E(\omega; p)$ has similar properties with those of $E(\omega; 1)$ or of $E(\omega; 2)$.

Theorem 1

If M is a complete Riemannian manifold and x, y are two points for which $d(x, y) = d$, then (1) the functional $E: \Omega \rightarrow R$, $p > 1$, has the minimum d^p , (2) the functional $E: \Omega \rightarrow R$, $p \in (-\infty, 0) \cup (0, 1)$, has the maximum d^p , and the value d^p is reached on the set of minimal geodesics from x to y .

Proof.

(1) Suppose $p > 1$. Applying Hölder's inequality

$$\int_0^1 |f g| dt \leq \left(\int_0^1 |f|^p dt \right)^{1/p} \left(\int_0^1 |g|^q dt \right)^{1/q}, \quad q = \frac{p}{p-1},$$

for $f(t) = \left\| \frac{d\omega}{dt} \right\|$, $g(t) = 1$, we deduce $L^p(\omega) \leq E(\omega; p)$.

If $f(t) = \text{const.}$, i.e., t is proportional to the arc-length then $L^p(\omega) = E(\omega; p)$ and conversely.

Let γ be a minimal geodesic from $x = \omega(0)$ to $y = \omega(1)$. We have $L(\gamma) = d$ and $E(\gamma; p) = L^p(\gamma) \leq L^p(\omega) \leq E(\omega; p)$. As $L^p(\gamma) = L^p(\omega)$ iff ω is a minimal geodesic, possibly reparametrized, and $L^p(\omega) = E(\omega; p)$ can hold only if the parameter t is proportional to the arc-length along ω , it follows that $E(\gamma; p) < E(\omega; p)$ unless ω is also a minimal geodesic.

(2) Now suppose $p \in (0, 1)$. Using Hölder's inequality

$$\int_0^1 |f g| dt \geq \left(\int_0^1 |f|^p dt \right)^{1/p} \left(\int_0^1 |g|^q dt \right)^{1/q}, \quad q = \frac{p}{p-1},$$

for $f(t) = \left\| \frac{d\omega}{dt} \right\|$, $g(t) = 1$, we obtain $L^p(\omega) \geq E(\omega; p)$.

By interchanging $p, f(t)$ respectively with $q, g(t)$ it follows that $L^p(\omega) \geq E(\omega; p)$ is also valid for $p \in (-\infty, 0)$.

The remainder of the proof is as in (1).

We shall now proceed to investigate which paths ω in Ω are critical points for the functional $E = E(\omega; p)$, $p \in \mathbb{R} - \{0\}$. We shall therefore denote $\bar{\alpha}$ = a variation of $\omega \in \Omega$, W_t = the associated vector field to $\bar{\alpha}$, $V_t = \frac{d\omega}{dt}$ = the velocity vector of ω , $A_t = \frac{D}{dt} \frac{d\omega}{dt}$ = the acceleration vector of ω , $\Delta_t(\|V\|^{p-2} V) = (\|V\|^{p-2} V)_{t+} - (\|V\|^{p-2} V)_t$ = the discontinuity of $\|V\|^{p-2} V$ at $t \in (0, 1)$.

Theorem 2

(First variation formula). Using the above hypotheses and notations we have

$$\begin{aligned} \frac{1}{p} \frac{dE(\bar{\alpha}(u))}{du} \Big|_{u=0} &= - \sum_t \langle W_t, \Delta_t(\|V\|^{p-2} V) \rangle - \\ &- \int_0^1 \|V_t\|^{p-4} \langle W_t, \|V_t\|^2 A_t + (p-2) \langle A_t V_t, V_t \rangle dt. \end{aligned}$$

Proof.

The same arguments as in [4, §12].

Corollary

The path ω is a critical point of E iff it is a geodesic.

Proof.

We remember that a path $\omega \in \Omega$ is called a geodesic iff it is C^∞ on $[0, 1]$ and the acceleration vector field $A_t = \frac{D}{dt} \frac{d\omega}{dt}$ is identically zero along ω .

Let $\omega \in \Omega$ be a geodesic. The first variation formula implies that ω is a critical point for E .

Conversely, let ω be a critical point for E and $0 = t_0 < t_1 < \dots < t_k = 1$ be the division of $[0, 1]$ so that $\omega|_{[t_{i-1}, t_i]}$ are differentiable. As there is a variation of ω with $W_t = f(t)(\|V_t\|^2 A_t + (p-2) \langle A_t, V_t \rangle V_t)$, where $f(t) > 0$ for $t \neq t_i$ and $f(t) = 0$ for $t = t_i$, the first variation formula gives

$$-\frac{1}{p} \frac{dE}{du}(0) = \int_0^1 \|V_t\|^{p-4} f(t) (\|V_t\|^2 A_t + (p-2) \langle A_t, V_t \rangle V_t) dt = 0.$$

This equality holds iff

$$\|V_t\|^2 A_t + (p-2) \langle A_t, V_t \rangle V_t = 0$$

If $p = 1$, this gives $A_t = \|V_t\|^{-2} \langle A_t, V_t \rangle V_t$, i.e., $\omega|_{[t_{i-1}, t_i]}$ are reparametrized geodesics.

If $p \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$, then the above relation implies $A_t = 0$, and hence $\omega|_{[t_{i-1}, t_i]}$ are geodesics. In order to prove that these are restrictions of a geodesic ω which unites x with y , we choose a variation such that $W(t_i) = \Delta_{t_i}(\|V\|^{p-2} V)$. This gives

$$-\frac{1}{p} \frac{dE}{du}(0) = \sum_{t_i} \|\Delta_{t_i}(\|V\|^{p-2} V)\|^2 = 0,$$

i.e., $\Delta_{t_i}(\|V\|^{p-2} V) = 0$, and so ω is differentiable of class C^1 , even at the points t_i . The Uniqueness Theorem for differential equations shows that ω is C^∞ everywhere, i.e., an unbroken geodesic.

§ 3. The second variation of E

Let γ be a critical point of the functional $E = E(\omega; p)$, $p \in R - \{0\}$, i.e., a geodesic, and W_1, W_2 two vector fields in $T_\gamma \Omega$. Let $\alpha: U \times [0, 1] \rightarrow M$, $(0, 0) \in U \subset R^2$, be a 2-parameter variation of γ so that

$$\alpha(0, 0, t) = \gamma(t), \quad \frac{\partial \alpha}{\partial u_1}(0, 0, t) = W_1(t), \quad \frac{\partial \alpha}{\partial u_2}(0, 0, t) = W_2(t),$$

and $\bar{\alpha}(u_1, u_2) \in \Omega$ be the path defined by $\bar{\alpha}(u_1, u_2)(t) = \alpha(u_1, u_2, t)$.

To show that the Hessian E_{**} is well defined the following Theorem is necessary.

Theorem

(Second variation formula). *Using the above hypotheses and notations we have*

$$\begin{aligned} \frac{1}{p} \frac{\partial^2 E}{\partial u_1 \partial u_2} (0, 0) = & - \sum_t \|V\|^{p-4} \langle W_2, \|V\|^2 \Delta_t \frac{DW_1}{dt} + \\ & + (p-2) \langle \Delta_t \frac{DW_1}{dt}, V \rangle V - \int_0^1 \|V\|^{p-4} \langle W_2, \|V\|^2 \left(\frac{D^2 W_1}{dt^2} + \right. \\ & \left. + R(V, W_1) V \right) \rangle V + (p-2) \langle V, \frac{D^2 W_1}{dt^2} + R(V, W_1) V \rangle V dt, \end{aligned}$$

where $V = \frac{d\gamma}{dt}$, $\Delta_t \frac{DW_1}{dt} = \frac{DW_1}{dt}(t^+) - \frac{DW_1}{dt}(t^-)$ means the jump in $\frac{DW_1}{dt}$ at one of its finitely many points of discontinuity in $(0, 1)$ and R is the curvature tensor field.

Proof.

According to the proof of Theorem 2, we have

$$\begin{aligned} \frac{1}{p} \frac{\partial E}{\partial u_2} = & - \sum_t \left\langle \frac{\partial \alpha}{\partial u_2}, \Delta_t \left(\left\| \frac{\partial \alpha}{\partial t} \right\|^{p-2} \frac{\partial \alpha}{\partial t} \right) \right\rangle - \\ & - \int_0^1 \left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{\frac{p}{2}-1} \frac{\partial \alpha}{\partial u_2}, \frac{D}{dt} \frac{\partial \alpha}{\partial t} dt - \\ & - (p-2) \int_0^1 \left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{\frac{p}{2}-2} \left\langle \frac{\partial \alpha}{\partial u_2}, \frac{\partial \alpha}{\partial t} \right\rangle \left\langle \frac{D}{dt} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle dt. \end{aligned}$$

Differentiating with respect to u_1 we find

$$\begin{aligned} \frac{1}{p} \frac{\partial^2 E}{\partial u_1 \partial u_2} = & - \sum_t \left\langle \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial u_2}, \Delta_t \left(\left\| \frac{\partial \alpha}{\partial t} \right\|^{p-2} \frac{\partial \alpha}{\partial t} \right) \right\rangle - \\ & - \sum_t \left\langle \frac{\partial \alpha}{\partial u_2}, \frac{D}{\partial u_1} \Delta_t \left(\left\| \frac{\partial \alpha}{\partial t} \right\|^{p-2} \frac{\partial \alpha}{\partial t} \right) \right\rangle - \\ & - (p-2) \int_0^1 \left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{\frac{p}{2}-2} \left\langle \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle \left\langle \frac{\partial \alpha}{\partial u_2}, \frac{D}{dt} \frac{\partial \alpha}{\partial t} \right\rangle dt - \end{aligned}$$

$$\begin{aligned}
& - \int_0^1 \left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{p/2-1} \left(\left\langle \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial t}, \frac{D}{\partial t} \frac{\partial \alpha}{\partial t} \right\rangle + \left\langle \frac{\partial \alpha}{\partial u_2}, \frac{D}{\partial u_1} \frac{D}{\partial t} \frac{\partial \alpha}{\partial t} \right\rangle \right) dt - \\
& - (p-2)(p-4) \int_0^1 \left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{p/2-3} \left\langle \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle \left\langle \frac{\partial \alpha}{\partial u_2}, \frac{\partial \alpha}{\partial t} \right\rangle < \\
& < \frac{D}{\partial t} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle dt - (p-2) \int_0^1 \left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{p/2-2} \left(\left\langle \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle + \right. \\
& + \left. \left\langle \frac{\partial \alpha}{\partial u_2}, \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial t} \right\rangle \right) \left\langle \frac{D}{\partial t} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle dt - (p-2) \int_0^1 \left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{p/2-2} \cdot \\
& \cdot \left\langle \frac{\partial \alpha}{\partial u_2}, \frac{\partial \alpha}{\partial t} \right\rangle \left(\left\langle \frac{D}{\partial u_1} \frac{D}{\partial t} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle + \left\langle \frac{D}{\partial t} \frac{\partial \alpha}{\partial t}, \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial t} \right\rangle \right) dt.
\end{aligned}$$

The second term can be transcribed using

$$\begin{aligned}
& \frac{D}{\partial u_1} \Delta_t \left(\left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{p/2-1} \frac{\partial \alpha}{\partial t} \right) = \Delta_t \frac{D}{\partial u_1} \left(\left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle^{p/2-1} \frac{\partial \alpha}{\partial t} \right) = \\
& = \Delta_t ((p-2) \left\| \frac{\partial \alpha}{\partial t} \right\|^{p-4} \left\langle \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle \frac{\partial \alpha}{\partial t} + \left\| \frac{\partial \alpha}{\partial t} \right\|^{p-2} \frac{D}{\partial u_1} \frac{\partial \alpha}{\partial t}).
\end{aligned}$$

Now, all we have to do is to evaluate $\frac{\partial^2 E}{\partial u_1 \partial u_2}$ at $(u_1, u_2) = (0, 0)$.

For this we take into account successively

$$\Delta_t \frac{\partial \alpha}{\partial t} = 0, \quad \frac{D}{\partial t} \frac{\partial \alpha}{\partial t} = 0, \quad (\gamma = \bar{\alpha}(0, 0) \text{ is a geodesic}),$$

$$\frac{D}{\partial u_1} V = \frac{D}{\partial t} \frac{\partial \alpha}{\partial u_1} = \frac{D}{dt} W_1,$$

$$\frac{D}{\partial u_1} \frac{D}{\partial t} V = \frac{D}{\partial t} \frac{D}{\partial u_1} V + R(V, W_1)V = \frac{D^2 W_1}{dt^2} + R(V, W_1)V.$$

This completes the proof.

The second variation formula and $\frac{\partial^2 E}{\partial u_1 \partial u_2} = \frac{\partial^2 E}{\partial u_2 \partial u_1}$ show that the following consequence is true.

Corollary

$E_{**}(W_1, W_2) = \frac{\partial^2 E}{\partial u_1 \partial u_2}(0, 0)$ is a well defined symmetric and bilinear function of W_1 and W_2 .

§4. The null space of the Hessian of E .

Let γ be a geodesic and $V = \frac{d\gamma}{dt}$ the tangent vector field of γ . A vector field J along γ which satisfies

$$\frac{D^2 J}{dt^2} + R(V, J)V = 0$$

is called a *Jacobi field* [3], [4]. The Jacobi equation has $2n$ linearly independent C^∞ -differentiable solutions, each being completely determined by $J(0)$ and $\frac{DJ}{dt}(0)$.

Let $p = \gamma(a)$ and $q = \gamma(b)$, $a \neq b$, be two points on the geodesic γ . These are called *conjugate* to each other along γ if there is a non-zero Jacobi field J which vanishes for $t = a$ and $t = b$. The dimension of the space of all Jacobi fields along γ which vanishes for $t = a$ and $t = b$ is called the *multiplicity* of p and q as conjugate points.

Now let γ be a geodesic in Ω . The set

$$N = \{W_1 | W_1 \in T_\gamma \Omega, E_{**}(W_1, W_2) = 0, \forall W_2 \in T_\gamma \Omega\}$$

is called the *null space* of the Hessian $E_{**} : T_\gamma \Omega \times T_\gamma \Omega \rightarrow R$ and $\dim N$ is called the *nullity* of E_{**} . Obviously $\dim N > 0$ iff E_{**} is degenerate.

Theorem 4

(1) Let $p \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$. Then $W_1 \in N$ iff it is a Jacobi field.

(2) Let $p = 1$ and $W_1 \in T_\gamma \Omega$. Then $W_1^\perp = W_1 - \|V\|^2 < W_1$, $V > V$ belongs to N iff it is a Jacobi field.

In other words, E_{**} is degenerate iff x and y are conjugate along γ . The nullity of E_{**} is equal to the multiplicity of x and y as conjugate points.

Proof.

(1) Let J be a Jacobi field which vanishes in $x = \gamma(0)$, $y = \gamma(1)$. Obviously $J \in T_\gamma \Omega$ and the second variation formula gives $E_{**}(J, W_2) = 0, \forall W_2$, i.e., $J \in N$.

Conversely, suppose that $W_1 \in N$. Choose a division $0 = t_0 < t_1 < \dots < t_k = 1$ of $[0, 1]$ so that the restriction of W_1 to $[t_{i-1}, t_i]$ be differentiable. We take W_2 of the form

$$W_2(t) = f(t) \left\{ \|V\|^2 \left(\frac{D^2 W_1}{dt^2} + R(V, W_1)V \right) + (p-2) \langle V, \frac{D^2 W_1}{dt^2} + R(V, W_1)V \rangle V \right\},$$

where $f(t) > 0$ for $t \neq t_i$ and $f(t) = 0$ for $t = t_i$. With these, the second variation formula implies

$$-\frac{1}{p} E_{**}(W_1, W_2) = \int_0^1 \|V\|^{p-4} f(t) \left\| \|V\|^2 \left(\frac{D^2 W_1}{dt^2} + R(V, W_1)V \right) + (p-2) \langle V, \frac{D^2 W_1}{dt^2} + R(V, W_1)V \rangle V \right\|^2 dt = 0.$$

It follows that

$$\|V\|^2 \left(\frac{D^2 W_1}{dt^2} + R(V, W_1)V \right) + (p-2) \langle V, \frac{D^2 W_1}{dt^2} + R(V, W_1)V \rangle V = 0$$

which (for $p \in \mathbb{R} - \{0, 1\}$) is equivalent to

$$\frac{D^2 W_1}{dt^2} + R(V, W_1)V = 0.$$

So $W_1|_{[t_{i-1}, t_i]}$, $i = 1, 2, \dots, k$, are Jacobi fields. To show that these are restrictions of a Jacobi field W defined on $[0, 1]$ we choose $W'_2 \in T_{\gamma}\Omega$ so that

$$W'_2(t_i) = \|V\|^2 \Delta_{t_i} \frac{DW_1}{dt} + (p-2) \langle \Delta_{t_i} \frac{DW_1}{dt}, V \rangle V, \quad i = 1, 2, \dots, k-1.$$

This selection gives

$$-\frac{1}{p} E_{**}(W_1, W'_2) = \sum_{i=1}^{k-1} \|V\|^{p-4} \left\| \|V\|^2 \Delta_{t_i} \frac{DW_1}{dt} + (p-2) \langle \Delta_{t_i} \frac{DW_1}{dt}, V \rangle V \right\|^2 = 0$$

and hence

$$\|V\|^2 \Delta_{t_i} \frac{DW_1}{dt} + (p-2) \langle \Delta_{t_i} \frac{DW_1}{dt}, V \rangle V = 0 \quad \text{or} \quad \Delta_{t_i} \frac{DW_1}{dt} = 0,$$

i.e., $\frac{DW_1}{dt}$ has no jumps. Having in mind that a solution of the Jacobi equation is completely determined by the vectors $W_1(t_i)$, $\frac{DW_1}{dt}(t_i)$ we deduce that $W_1|_{[t_{i-1}, t_i]}$ are restrictions of the C^∞ Jacobi field W_1 on $[0, 1]$.

(2) Let $p=1$ and $W_1, W_2 \in T_\gamma \Omega$. As $\frac{DV}{dt}=0$, $\langle R(V, W_1)V, V \rangle = 0$ and $R(V, W_1)V = R(V, W_1^\perp)V$, the second variation formula takes the form

$$\begin{aligned} \frac{\partial^2 E}{\partial z_1 \partial u_2}(0, 0) = & - \sum_i \langle W_2, \Delta_i \frac{DW_1^\perp}{dt} \rangle - \\ & - \int_0^1 \|V\|^{-2} \langle W_2, \frac{D^2 W_1^\perp}{dt^2} + R(V, W_1^\perp)V \rangle dt. \end{aligned}$$

This formula and trivial modifications in arguments of (1) or of [3, Ch. VIII, 5)] show that the assertion (2) is true.

Because along γ we have only finitely many linearly independent Jacobi fields, the nullity of E_{**} is finite.

Other properties of the functional E suggested by [1]–[7] will be studied within another paper.

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Polytechnical Institute Bucharest

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... in the following manner: let a solution of the Jacobi equation be sought in the form $W = \sum_{i=1}^n W_i e_i$, where W_i are functions of t . Substituting this into the Jacobi equation and using the orthogonality of the e_i , we obtain a system of n linear equations for the W_i . This system can be written in matrix form as $W' = A W$, where A is a matrix whose elements are functions of t . The solution of this system is given by $W = e^{At} W(0)$, where $W(0)$ is the initial value of W at $t=0$.

$$W' = A W$$
$$W = e^{At} W(0)$$

This formula and the initial conditions in paragraph (1) on p. 13 of [1] show that the solution is unique. Because of this, we have only to verify that the function W defined by (1) satisfies the Jacobi equation. This is done by substituting (1) into the Jacobi equation and using the orthogonality of the e_i . Other properties of the function W are suggested by (1) and (2) which are omitted for brevity.

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INTEGRAL EQUATIONS INVOLVING THE G-FUNCTION AS KERNEL

R. U. VERMA

ABSTRACT

The present study deals with certain integral equations whose solutions can be found by using the technique of L and L^{-1} operators. L and L^{-1} denote the Laplace transform and its inverse. Later some special cases are derived from the main result which may prove of general interest.

1. Introduction

In recent works of Fox [3], certain integral equations have been solved by means of L and L^{-1} operators. These solutions need a comprehensive tables [2] of Laplace transforms and its inverse, which exist. The object of this paper is to find the solution of the integral equation involving the G -function as kernel by L and L^{-1} operators.

The integral equations of the type

$$(1.1) \quad \int_0^{\infty} (x|t)^{\alpha} G_{1,2}^{2,0} (xt|_{b,c}^a) f(t) dt = g(x), \quad (x > 0),$$

where g is given and f is the unknown function to be determined.

In (1.1), $G_{1,2}^{2,0} (x) [1, p. 207]$ denotes

$$(1.2) \quad G_{1,2}^{2,0} (x|_{b,c}^a) = (2xi)^{-1} \int_c^{\infty} M_{2,1} (s) x^{-s} ds,$$

where

$$(1.3) \quad M_{2,1} (s) = \Gamma(b+s) \Gamma(c+s) \{\Gamma(a+s)\}^{-1},$$

$$(1.4) \quad \mathcal{M} [G_{1,2}^{2,0} (x)] = M_{2,1} (s),$$

where $\mathcal{M}$ denotes the Mellin transform.

We recall the following theorem of Fox [3] which will be needed in developing the solution.

Theorem. If,

$$(i) \quad \alpha > 0, 1/2\alpha + \beta > 0, t > 0;$$

$$(ii) \quad s = \sigma + i\mu, \sigma \text{ and } \mu \text{ both real; } F(s) \in \mathcal{L}(1/2 - i\infty, 1/2 + i\infty);$$

then

$$(1.5) \quad L^{-1} \left\{ (2\pi i)^{-1} \int_C \Gamma(\alpha s + \beta) F(s) t^{-\alpha s - \beta} ds \right\} = (2\pi i)^{-1} \int_C F(s) x^{\alpha s + \beta - 1} ds,$$

where, for both integrals, the contour C may be the line $\sigma = 1/2$, a line parallel to the imaginary axis in the complex s -plane.

Next we recall the Mellin-Parseval Theorem : If,

$$(1.6) \quad \mathcal{M}[h(u)] = H(s) \text{ and } \mathcal{M}[f(u)] = F(s),$$

then

$$(1.7) \quad \int_0^\infty h(u) f(u) du = (2\pi i)^{-1} \int_C H(s) F(1-s) ds,$$

where C is a suitable contour in s -plane and $\mathcal{M}$ denotes the Mellin transform.

2. Solution of (1.1) as an integral equation

Theorem. If,

- (i) $b > 0, c > 0, a > -1/2$;
- (ii) $f(x) \in \mathcal{L}_2(0, \infty)$;
- (iii) $s^{b+c-a} F(1-s) \in \mathcal{L}(1/2 - i\infty, 1/2 + i\infty)$;
- (iv) $F(1-s) \in \mathcal{L}(1/2 - i\infty, 1/2 + i\infty)$ and
- (v) $y^{-1/2} f(y) \in \mathcal{L}(0, \infty)$, where $f(y)$ is of bounded variation near the

point $y = t$, then the solution of (1.1), as an integral equation of $f(t)$ is :

$$(2.1) \quad f(t) = t^{v-b} L^{-1} \{ x^{a-b} L[t^{a-c} L^{-1} \{ x^{-c-v} g(x) \}] \}.$$

PROOF. In order to solve (1.1), we first apply (1.7) to the left hand side of (1.1). Then, we try to eliminate the three gamma functions from the integrand of the resulting equation by using the method of L^{-1} and L operators. Using the asymptotic expansion of the gamma function [6, p. 272] along the line $s = 1/2 + it$, for large $|t|$, it can be shown that we can use equation (1.5) to eliminate the gamma function from the integrand by L^{-1} operator. Then, again we can apply L operator to eliminate the gamma function from the denominator. Thus, we can arrive at the result (2.1).

3. Applications

(A) By taking $\nu = 0$, $a = \mu - k + 1/2$, $b = 2\mu$, $c = 0$, our theorem leads to

COROLLARY. 1. If,

(i) $\mu > 0$, $\mu - k > -1$;

(ii) $f(x) \in \mathcal{L}_2(0, \infty)$;

(iii) $s^{\mu+k-1/2} F(1-s) \in \mathcal{L}(1/2 - i\infty, 1/2 + i\infty)$;

(iv) $F(1-s) \in \mathcal{L}(1/2 - i\infty, 1/2 + i\infty)$ and

(v) $y^{1/2} f(y) \in \mathcal{L}(0, \infty)$, where $f(y)$ is of bounded variation near the point $y = t$, then the solution of [8]

$$(3.1) \quad \int_0^\infty [(xt)^{\mu-1/2} e^{-(1/2)xt} W_{k,\mu}(xt)] f(t) dt = g(x), \quad (x > 0),$$

as an integral equation is

$$(3.2) \quad f(t) = t^{-2\mu} L^{-1} \{x^{-\mu-k+1/2} L[t^{\mu-k+1/2} L^{-1} \{g(x)\}]\},$$

$W_{k,\mu}(z)$ being Whittaker's function [6, p. 334]. This solution has been obtained recently by Fox [4, (12)].

(B) Again, if we write $t^{-\nu} f(t) = f_1(t)$ and $t^{-\nu} g(t) = g_1(t)$, the equation (1.1) reduces to the integral equation:

$$(3.3) \quad \int_0^\infty G_{1,2}^{2,0}(xt|_{b,c}^a) f_1(t) dt = g_1(x), \quad (x > 0),$$

and whose solution can be written under the similar conditions as in the main theorem as

$$(3.4) \quad f_1(t) = t^{-b} L^{-1} \{x^{a-b} L[t^{a-c} L^{-1} \{x^{-c} g_1(x)\}]\}.$$

(C) With $\mu + k = 1/2$ in (3.1) we get

$$(3.5) \quad L[f(t)] = g(x)$$

Since $L^{-1}L = 1$, (3.2) reduces to

$$(3.3) \quad f(t) = L^{-1} \{g(x)\}.$$

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University of Cape Coast, Cape Coast, Ghana

INITIAL ALGEBRA SEMANTICS FOR KNUTH SYSTEMS WITH INHERITED ATTRIBUTES

MARIA ZAMFIR

I. Introduction

In this paper we give an algebraic formulation of „Knuth semantics” including inherited attributes [2], [3], using the „Initial algebra semantics” [1]. We also give as a nontrivial application an initial algebra semantics for Knuth's language TURINGOL [2], [3].

This approach provides, apparently for the first time, a rigorous definition of evaluation for Knuth semantics. The main idea is to introduce variables for the values of the inherited attributes.

The synthesized attributes are terms in these variables, and evaluation proceeds by substitution. For example, a precise definition of binary notation for numbers is done by the following Knuth system:

<i>Context-free grammar G</i>	<i>Semantic rules</i>
$P_1 B \rightarrow 0$	$v(B) = 0$
$P_2 B \rightarrow 1$	$v(B) = 2^{s(B)}$
$P_3 L \rightarrow B$	$v(L) = v(B) \quad , \quad l(L) = 1$ $s(B) = s(L)$
$P_4 L_1 \rightarrow L_2 B$	$v(L_1) = v(L_2) + v(B) \quad , \quad P(L_1) = P(L_2) + 1$ $s(B) = s(L_1) \quad \quad \quad s(L_2) = s(L_1) + 1$
$P_5 N \rightarrow L$	$v(N) = v(L) \quad , \quad s(L) = 0$
$P_6 N \rightarrow L_1 \cdot L_2$	$v(N) = v(L_1) + v(L_2)$ $s(L_1) = 0 \quad \quad \quad , \quad s(L_2) = -P(L_2)$

In this example v and P are synthesized attributes, and s is an inherited attribute with which we associate the variable X . We also associate with every production P_i , $1 \leq i \leq 6$, an operation, p_i , $1 \leq i \leq 6$:

$$p_1 = \langle 0, 1 \rangle$$

$$p_2 = \langle 2^X, 1 \rangle$$

$$p_3(\langle \alpha, \beta \rangle) = \langle \alpha, \beta \rangle$$

$$p_4(\langle \alpha, \beta \rangle, \langle \alpha_2, \beta_2 \rangle) = \langle (\alpha_1 \xleftarrow{X} X + 1) + \alpha_2, \beta_1 + 1 \rangle$$

$$p_5(\langle \alpha, \beta \rangle) = \langle \alpha \xleftarrow{X} 0, \beta \rangle$$

$$p_6(\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle) = \langle \alpha_1 \xleftarrow{X} 0 + \alpha_2 \leftarrow -\beta_2, \beta_1 \rangle$$

Semantic rules are added to G in the following manner :

To each symbol $X \in V$ we associate a finite set $A(X)$ of attributes ; $A(X)$ is partitioned into two disjoint sets, the synthesized attributes $A_0(X)$ and the inherited attributes $A_1(X)$. For regularity, we suppose that $A_0(X)$ and $A_1(X)$ are the same for every $X \in V$: $A_0(X) = \{\alpha_1, \dots, \alpha_l\}$ and $A_1(X) = \{\beta_1, \dots, \beta_s\}$ except $A_1(S)$ which is empty and $A_0(X)$ and $A_1(X)$ which are also empty if X is a terminal symbol. Each attribute α_i or β_k , $1 \leq i \leq l$, $1 \leq k \leq s$, in $A(X)$ has a (possibly infinite) set of possible values V_i or V_{k+1} from which one value will be selected (by means of the semantic rules) for each appearance of X in a derivation tree. Let $\mathcal{P}$ consist of m productions, and let p -th production be

$$P : X_{p0} \rightarrow u_{p0} X_{p1} u_{p1} X_{p2} \dots X_{pn_p} u_{pn_p}$$

where $n_p \geq 0$, $X_{p0} \in N$, $X_{pj} \in V$, $u_{pj} \in (V - N)^*$ for $1 \leq j \leq n_p$

The semantic rules associated with every production are functions „ f ” defined for all synthesized attributes and „ g ” defined for all inherited attributes, at all nodes, in any conceivable derivation tree. Because all actions on global quantities can be reduced to a sequence of semantic rules relating a synthesized and an inherited attribute, we don't consider them a special case.

For production P we have $f_{p1}, \dots, f_{pl}, g_{p11}, \dots, g_{p1s}, \dots, g_{pn_p1}, \dots, g_{pn_p s}$

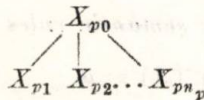
Each f_{pi} and g_{pjks} , $1 \leq i \leq l$, $1 \leq j \leq n$, $1 \leq k \leq s$, is a mapping of $V_1 \times V_2 \times \dots \times V_{l+s}$ into V_i resp. V_{k+1} , $1 \leq i \leq l$, $1 \leq k \leq s$. Now, for every $\omega \in V^*$ define $nt(\omega)$ to be the string of nonterminals in ω .

We make G into an N -sorted operator domain by

$$GX_{p1} \dots X_{pn_p}, X_{p0} = \{P \in \mathcal{P} \mid P = \langle X_{p0}, u_{p0} X_{p1} u_{p1} \dots X_{pn_p} u_{pn_p} \rangle \text{ and}$$

$$nt(u_{p0} X_{p1} u_{p1} \dots X_{pn_p} u_{pn_p}) = x_{p1} \dots X_{pn_p}\}$$

Thus the production P is an operator symbol of type $\langle X_{p1} \dots X_{pn_p}, X_{p0} \rangle$ and rank n_p . We get in this way the initial G -algebra T_G which has carriers $T_{G, X_{p0}}$ which are derivation trees for derivations in G from X_{p0} .



We associate a variable Y_k , $1 \leq k \leq s$ with every inherited attribute β_k , $1 \leq k \leq s$, $Y = \{Y_1, \dots, Y_s\}$ being disjoint from V . Every synthesized attribute will become a term in s variables (the inherited

attribute will become a term in s variables (the inherited attributes) and f_{pi} and g_{pjk} will be functions of Y_k , $1 \leq i \leq l$, $1 \leq j \leq n_p$, $1 \leq k \leq s$. Let $V = V_1 \cup V_2 \cup \dots \cup V_{l+s}$. Also, let $\Omega_0 = V \cup Y$ and $\Omega_r = \{F_P \mid \# \# nt(P)\} r, r > 0$, where

$$F_P(\langle v_1^1, v_2^1, \dots, v_l^1 \rangle, \langle v_1^2, v_2^2, \dots, v_l^2 \rangle, \dots, \langle v_1^r, v_2^r, \dots, v_l^r \rangle) = \\ = \langle f_{pi}(v_1^1 \leftarrow_Y g_{p1}, \dots, v_l^1 \leftarrow_Y g_{pr}), \dots, f_{pi}(v_1^1 \leftarrow_Y g_{p1}, \dots, v_l^1 \leftarrow_Y g_{pr}) \rangle$$

where $g_{pj} = \langle g_{pj1}, \dots, g_{pjs} \rangle$, $1 \leq j \leq r$, and $t \leftarrow_Y t'$ is the result of substituting t'_r for Y_r in t , $t' = \langle t_1, \dots, t_r, \dots, t_s \rangle$.

Now, we define a G -algebra A with each carrier $A_N = (T_\Omega)^l$, l being the number of synthesized attributes, and for every production P the operation F_P as above. Since T_G is the initial G -algebra [1], there is an unique G -homomorphism $h_A: T_G \rightarrow A$ which represents the Knuth semantics. This represents a rigorous definition of evaluation of Knuth systems.

III. Turingol

As a nontrivial illustration we give an algebraic semantics for TURINGOL as defined in [2], using some modifications suggested in [3]. These modifications were necessary for making the semantic rules well defined, the function „newsymbol” having different values for different order of passing of the tree. We have:

start — inherited attribute

follow, *index*, *text*, — synthesized attributes

symbol, *label*, *alphabet*, *states* — global attributes

We associate with every inherited attributes a variable, thus: W for *start*

SY for the inherited component of the global variable *symbol*

L for the inherited component of the global variable *label*

Σ for the inherited component of the global variable *alphabet*

Q for the inherited component of the global variable *states*

The grammar G

The semantic rules

$P_{1.1} \quad A \rightarrow a$

$\text{text}(A) = a.$

.....

.....

$P_{1.26} \quad A \rightarrow z$

$\text{text}(A) = z.$

$P_{2.1} \quad I \rightarrow A$

$\text{text}(I) = \text{text}(A).$

$P_{2.2} \quad I \rightarrow IA$

$\text{text}(I) = \text{text}(I) \text{text}(A).$

$P_{3.1}$	$D \rightarrow \text{tape}$ <i>alphabet</i> <i>is I</i>	<i>define</i> symbol (text (I)) = 1; <i>include</i> symbol (text (I)) in Σ ; index (D) = 1.
$P_{3.2}$	$D_1 \rightarrow D_2, I$	<i>define</i> symbol (text (I)) = index (D_1); <i>include</i> symbol (text (I)) in Σ ; index (D_1) = index (D_2) + 1.
$P_{4.1}$	$S \rightarrow \text{print } ,,I''$	<i>define</i> δ (start (S), s) = (symbol (text (I)), 0, follow (S)) for all $s \in \Sigma$; follow (S) = start (S) + 1; <i>include</i> follow (S) in Q.
$P_{4.2}$	$S \rightarrow \text{move O}$ <i>one square</i>	<i>define</i> δ (start (S), s) = (s, d (0)), follow (S)) for all $s \in \Sigma$; follow (S) = start (S) + 1; <i>include</i> follow (S) in Q.
$P_{4.2.1}$	$O \rightarrow \text{left}$	d(O) = - 1.
$P_{4.2.2}$	$O \rightarrow \text{right}$	d(O) = + 1.
$P_{4.3}$	$S \rightarrow \text{go to I}$	<i>define</i> δ (start (S), s) = (s, 0, label (text (I)) for all $s \in \Sigma$; follow (S) = start (S) + 1; <i>include</i> follow (S) in Q.
$P_{4.4}$	$S \rightarrow$	follow (S) = start (S).
$P_{5.1}$	$S_1 \rightarrow \text{if the tape}$ <i>symbol is ,,I''</i> <i>then S₂</i>	<i>define</i> δ (start (S_1), s) = (s, 0, follow (S_2)) for all $s \in \Sigma$ -symbol (text (I)); <i>define</i> δ (start (S_1), s) = (s, 0, start (S_2)) for s = symbol (text (I)); start (S_2) = start (S_1) + 1; follow (S_1) = follow (S_2); <i>include</i> start (S_2) in Q.
$P_{5.2}$	$S_1 \rightarrow I : S_2$	<i>define</i> label (text (I)) = start (S_1); start (S_2) = start (S_1); follow (S_1) = follow (S_2).
$P_{5.3}$	$S \rightarrow \{L\}$	start (L) = start (S); follow (S) = follow (L).
$P_{6.1}$	$L \rightarrow S$	start (S) = start (L); follow (L) = follow (S).
$P_{6.2}$	$L_1 \rightarrow L_2; S$	start (L_2) = start (L_1); start (S) = follow (L_2); follow (L_1) = follow (S).
P_7	$P \rightarrow D; L.$	start (L) = 1; <i>include</i> 1 in Q; stop = follow (L).

$$\begin{array}{lcl}
p_{1.1} & = \langle a, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W, \{\emptyset\}, \{\emptyset\} \rangle \\
p_{1.2} & = \langle b, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W, \{\emptyset\}, \{\emptyset\} \rangle \\
\vdots & & \\
p_{1.26} & = \langle z, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W, \{\emptyset\}, \{\emptyset\} \rangle \\
p_{4.2.1} & = \langle -1, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W, \{\emptyset\}, \{\emptyset\} \rangle \\
p_{4.2.2} & = \langle 1, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W, \{\emptyset\}, \{\emptyset\} \rangle \\
p_{4.4} & = \langle \Lambda, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W, \{\emptyset\}, \{\emptyset\} \rangle \\
\left. \begin{array}{l} p_{2.1} \\ p_{2.2} \\ p_{3.1} \\ p_{3.2} \\ p_{4.1} \\ p_{4.2} \\ p_{4.3} \\ p_{5.1} \\ p_{5.2} \\ p_{5.3} \\ p_{6.1} \\ p_{6.2} \\ p_7 \end{array} \right\} & \text{constants} & \\
p_{2.1} & \langle \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \rangle = \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \\
p_{2.2} & \langle \langle t_1, i_1, sy_1, l_1, \varepsilon_1, \omega_1, \delta_1, q_1 \rangle, \langle t_2, i_2, sy_2, l_2, \varepsilon_2, \omega_2, \delta_2, q_2 \rangle \rangle \\
& = \langle t_1 t_2, i_1, sy_1, l_1, \varepsilon_1, \omega_1, \delta_1, q_1 \rangle \\
p_{3.1} & \langle \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \rangle = \langle t, 1, sy \cup \{(t, 1)\}, l, \varepsilon \cup \{1\}, \omega, \delta, q \rangle \\
p_{3.2} & \langle \langle t_1, i_1, sy_1, l_1, \varepsilon_1, \omega_1, \delta_1, q_1 \rangle, \langle t_2, i_2, sy_2, l_2, \varepsilon_2, \omega_2, \delta_2, q_2 \rangle \rangle \\
& = \langle t_2, i_1 + 1, sy_1 \cup \{(t_2, i_1 + 1)\}, l_2, \varepsilon_1 \cup \{i_1 + 1\}, \omega_2, \delta_2, q_2 \rangle \\
p_{4.1} & \langle \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \rangle = \langle t, i, sy, l, \varepsilon, \omega, \leftarrow_W W + 1, \delta \cup \\
& \quad \cup \{(W, s, SY(t), \\
& \quad 0, \omega \leftarrow_W W + 1 \text{ for all } s \in \Sigma\}, q \cup \{\omega \leftarrow_W W + 1\} \rangle \\
p_{4.2} & \langle \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \rangle = \langle t, i, sy, l, \varepsilon, \omega \leftarrow_W W + 1, \delta \cup \\
& \quad \cup \{(W, s, s, t, \omega \leftarrow_W W + 1) \text{ for all } s \in \Sigma\}, q \cup \{\omega \leftarrow_W W + 1\} \rangle \\
p_{4.3} & \langle \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \rangle = \langle t, i, sy, l, \varepsilon, \omega \leftarrow_W W + 1, \delta \cup \\
& \quad \cup \{(W, s, s, 0, L(t)) \text{ for all } s \in \Sigma\}, q \cup \{\omega \leftarrow_W W + 1\} \rangle \\
p_{5.1} & \langle \langle t_1, i_1, sy_1, l_1, \varepsilon_1, \omega_1, \delta_1, q_1 \rangle, \langle t_2, i_2, sy_2, l_2, \varepsilon_2, \omega_2, \delta_2, q_2 \rangle \rangle \\
& = \langle t_1, i_1, sy_1, l_2 \leftarrow_W W + 1, \varepsilon_1, \omega_2 \leftarrow_W W + 1, \{(W, s, 0, \omega_2) \text{ for all } \\
& \quad s \in \Sigma - SY(t_1)\} \cup \{(W, s, s, 0, W + 1) \text{ for } s = SY(t_1)\} \cup \\
& \quad \cup \delta_2 \leftarrow_W W + 1 \{W + 1\} \cup q_2 \leftarrow_W W + 1 \rangle \\
p_{5.2} & \langle \langle t_1, i_1, sy_1, l_1, \varepsilon_1, \omega_1, \delta_1, q_1 \rangle, \langle t_2, i_2, sy_2, l_2, \varepsilon_2, \delta_2, q_2 \rangle \rangle \\
& = \langle t_1, i_1, sy_1, \{(t_1, W)\} \cup l_2, \varepsilon_1, \omega_2, \delta_2, q_2 \rangle \\
p_{5.3} & \langle \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \rangle = \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \\
p_{6.1} & \langle \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \rangle = \langle t, i, sy, l, \varepsilon, \omega, \delta, q \rangle \\
p_{6.2} & \langle \langle t_1, i_1, sy_1, l_1, \varepsilon_1, \omega_1, \delta_1, q_1 \rangle, \langle t_2, i_2, sy_2, l_2, \varepsilon_2, \omega_2, \delta_2, q_2 \rangle \rangle \\
& = \langle t_2, i_2, sy_2, l_1 \cup l_2 \leftarrow_W \omega_1, \varepsilon_2, \omega_2 \leftarrow_W \omega_1, \delta_1 \cup \delta_2 \leftarrow_W \omega_1, q_1 \cup q_2 \leftarrow_W \omega_1 \rangle \\
p_7 & \langle \langle t_1, i_1, sy_1, l_1, \varepsilon_1, \omega_1, \delta_1, q_1 \rangle, \langle t_2, i_2, sy_2, l_2, \varepsilon_2, \omega_2, \delta_2, q_2 \rangle \rangle \\
& = \langle t_1, i_1, sy_1, l_2 \leftarrow_W 1, \varepsilon_1, \omega_2 \leftarrow_W 1 \equiv stop, \delta_2 \xrightarrow{sy_1} \xrightarrow{sy} \xrightarrow{L} l_2, q_2 \leftarrow_W 1 \rangle
\end{array}$$

$$\begin{aligned}
p_{6.1}(p_{4.1}(p_{2.2}(\dots, \text{point}))) &= \langle \text{point}, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+1, \{(W, s, SY(\text{point}), 0, W+1) \text{ for all } s \in \Sigma\}, \{W+1\} \rangle \\
p_{4.3}(p_{2.2}(\dots, \text{carry} \dots)) &= \langle \text{carry}, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+1, \{(W, s, s, 0, L(\text{carry})) \text{ for all } s \in \Sigma\}, \{W+1\} \rangle \\
p_{6.2}(p_{6.1} \& p_{4.3}) &= \langle \text{carry}, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+2, \{(W, s, SY(\text{point}), 0, W+1) \text{ for all } s \in \Sigma, (W+1, s, s, 0, L(\text{carry})) \text{ for all } s \in \Sigma\}, \{W+1, W+2\} \rangle \\
p_{6.1}(p_{4.1}(p_{2.2}(\dots, \text{zero}))) &= \langle \text{zero}, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+1, \{(W, s, SY(\text{zero}), 0, W+1) \text{ for all } s \in \Sigma\}, \{W+1\} \rangle \\
p_{4.2}(p_{4.2.1}) &= \langle -1, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+1, \{(W, s, s, -1, W+1) \text{ for all } s \in \Sigma\}, \{W+1\} \rangle \\
p_{5.2}(p_{2.2} \& p_{4.2}) &= \langle \text{carry}, 0, \{\emptyset\}, \{(\text{carry}, W)\}, \{\emptyset\}, W+1, \{(W, s, s, -1, W+1) \text{ for all } s \in \Sigma\}, \{W+1\} \rangle \\
p_{6.2}(p_{6.1} \& p_{5.2}) &= \langle \text{carry}, 0, \{\emptyset\}, \{(\text{carry}, W+1)\}, \{\emptyset\}, W+2, \{(W, s, SY(\text{zero}), 0, W+1) \text{ for all } s \in \Sigma, (W+1, s, s, -1, W+2) \text{ for all } s \in \Sigma\}, \{W+1, W+2\} \rangle \\
p_{4.3}(p_{2.2}(\dots, \text{test} \dots)) &= \langle \text{test}, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+1, \{(W, s, s, 0, L(\text{test})) \text{ for all } s \in \Sigma\}, \{W+1\} \rangle \\
p_{6.2}(p_{6.2} \& p_{4.3}) &= \langle \text{test}, 0, \{\emptyset\}, \{(\text{carry}, W+1)\}, \{\emptyset\}, W+3, \{(W, s, SY(\text{zero}), 0, W+1) \text{ for all } s \in \Sigma, (W+1, s, s, -1, W+2) \text{ for all } s \in \Sigma, (W+2, s, 0, L(\text{test})) \text{ for all } s \in \Sigma\}, \{W+1, W+2, W+3\} \rangle = p_{5.3} \\
p_{5.1}(p_{2.2} \& p_{5.3}) &= \langle \text{one}, 0, \{\emptyset\}, \{(\text{carry}, W+2)\}, \{\emptyset\}, W+4, \{(W, s, s, 0, W+3) \text{ for all } s \in \Sigma - SY(\text{one}), (W, s, s, 0, W+1) \text{ for } s = SY(\text{one}), (W+1, s, SY(\text{zero}), 0, W+2) \text{ for all } s \in \Sigma, (W+2, s, s, -1, W+3) \text{ for all } s \in \Sigma, (W+3, s, s, 0, L(\text{test})) \text{ for all } s \in \Sigma\}, \{W+1, W+2, W+3, W+4\} \rangle \\
p_{5.2}(p_{2.2} \& p_{5.1}) &= \langle \text{test}, 0, \{\emptyset\}, \{(\text{test}, W), (\text{carry}, W+2)\}, \{\emptyset\}, W+4, \{(W, s, s, 0, W+3) \text{ for all } s \in \Sigma - SY(\text{one}), (W, s, s, 0, W+1) \text{ for } s = SY(\text{one}), (W+1, s, SY(\text{zero}), 0, W+2) \text{ for all } s \in \Sigma, (W+2, s, s, -1, W+3) \text{ for all } s \in \Sigma, (W+3, s, s, 0, L(\text{test})) \text{ for all } s \in \Sigma\}, \{W+1, W+2, W+3, W+4\} \rangle
\end{aligned}$$

$$\begin{aligned}
p_{6.2}(p_{6.2} \& p_{5.2}) &= \langle \text{test}, 0, \{\emptyset\}, \{(\text{test}, W+2), (\text{carry}, W+4)\}, \{\emptyset\}, W+6 \{ (W, s, SY(\text{point}), 0, W+1) \text{ for all } s \in \Sigma, (W+1, s, 0, L(\text{carry})) \text{ for all } s \in \Sigma, (W+2, s, s, 0, W+5) \text{ for all } s \in \Sigma - SY(\text{one}), (W+2, s, s, 0, W+3) \text{ for } s = SY(\text{one}), (W+3, s, SY(\text{zero}), 0, W+4) \text{ for all } s \in \Sigma, (W+4, s, s, -1, W+5) \text{ for all } s \in \Sigma, (W+5, s, s, 0, L(\text{test})) \text{ for all } s \in \Sigma\}, \{W+1, W+2, W+3, W+4, W+5, W+6\} \rangle \\
P_{4.1}(p_{2.2}(\dots, \text{one} \dots)) &= \langle \text{one}, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+1, \{ (W, s, SY(\text{one}), 0, W+1) \text{ for all } s \in \Sigma\}, \{W+1\} \rangle \\
p_{6.2}(p_{6.2} \& p_{4.1}) &= \langle \text{one}, 0, \{\emptyset\}, \{\emptyset\}, \{(\text{test}, W+2), (\text{carry}, W+4)\}, \{\emptyset\}, W+7, \{ (W, s, SY(\text{point}), 0, W+1) \text{ for all } s \in \Sigma, (W+1, s, s, 0, L(\text{carry})) \text{ for all } s \in \Sigma, (W+2, s, s, 0, W+5) \text{ for all } s \in \Sigma - SY(\text{one}), (W+2, s, s, 0, W+3) \text{ for } s = SY(\text{one}), (W+3, s, SY(\text{zero}), 0, W+4) \text{ for all } s \in \Sigma, (W+4, s, s, -1, W+5) \text{ for all } s \in \Sigma, (W+5, s, s, 0, L(\text{test})) \text{ for all } s \in \Sigma, (W+6, s, SY(\text{one}), 0, W+7) \text{ for all } s \in \Sigma\}, \{W+1, W+2, W+3, W+4, W+5, W+6, W+7\} \rangle \\
p_{4.2}(p_{4.2.2}) &= \langle +1, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+1, \{ (W, s, s, 1, W+1) \}, \{W+1\} \rangle \\
p_{5.2}(p_{2.2} \& p_{4.2}) &= \langle \text{realign}, 0, \{\emptyset\}, \{(\text{realign}, W)\}, \{\emptyset\}, W+1, \{ (W, s, s, 1, W+1) \text{ for all } s \in \Sigma\}, \{W+1\} \rangle \\
p_{6.2}(p_{6.2} \& p_{5.2}) &= \langle \text{realign}, 0, \{\emptyset\}, \{(\text{test}, W+2), (\text{carry}, W+4), (\text{realign}, W+7)\}, \{\emptyset\}, W+8, \{ (W, s, SY(\text{point}), 0, W+1) \text{ for all } s \in \Sigma, (W+1, s, s, 0, L(\text{carry})) \text{ for all } s \in \Sigma, (W+2, s, s, 0, W+5) \text{ for all } s \in \Sigma - SY(\text{one}), (W+2, s, s, 0, W+3) \text{ for } s = SY(\text{one}), (W+3, s, SY(\text{zero}), 0, W+4) \text{ for all } s \in \Sigma, (W+4, s, s, -1, W+5) \text{ for all } s \in \Sigma, (W+5, s, s, 0, L(\text{test})) \text{ for all } s \in \Sigma, (W+6, s, SY(\text{one}), 0, W+7) \text{ for all } s \in \Sigma, (W+7, s, s, 1, W+8) \text{ for all } s \in \Sigma\}, \{W+1, W+2, W+3, W+4, W+5, W+6, W+7, W+8\} \rangle
\end{aligned}$$

$$\begin{aligned}
p_{5.1} (p_{2.2} \& p_{4.3}) &= \langle \text{zero}, 0, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}, W+2, \{(W, s, s, 0, W+2) \text{ for } s \in \Sigma - SY(\text{zero}), (W, s, s, 0, W+1) \text{ for } s = SY(\text{zero}), (W+1, s, s, 0, L(\text{realign})) \text{ for all } s \in \Sigma\}, \{W+1, W+2\} \rangle \\
p_{6.2} (p_{6.2} \& p_{5.1}) &= \langle \text{zero}, 0, \{\emptyset\}, \{(\text{test}, W+2), (\text{carry}, W+4), (\text{realign}, W+7)\}, \{\emptyset\}, W+10, \{(W, s, SY(\text{point}), 0, W+1) \text{ for all } s \in \Sigma, (W+1, s, s, 0, L(\text{carry})) \text{ for all } s \in \Sigma, (W+2, s, s, 0, W+5) \text{ for all } s \in \Sigma - SY(\text{one}), (W+2, s, s, 0, W+3) \text{ for } s = SY(\text{one}), (W+3, s, SY(\text{zero}), 0, W+4) \text{ for all } s \in \Sigma, (W+4, s, s, -1, W+5) \text{ for all } s \in \Sigma, (W+5, s, s, 0, L(\text{test}), \text{ for all } s \in \Sigma, (W+6, s, SY(\text{one}), 0, W+7) \text{ for all } s \in \Sigma, (W+7, s, s, 1, W+8) \text{ for all } s \in \Sigma, (W+8, s, s, 0, W+10) \text{ for } s \in \Sigma - SY(\text{zero}), (W+8, s, s, 0, W+9) \text{ for } s = SY(\text{zero}), (W+9, s, s, 0, L(\text{realign})) \text{ for all } s \in \Sigma\}, \{W+1, W+2, W+3, W+4, W+5, W+6, W+7, W+8, W+9, W+10\} \rangle \\
p_7 (p_{3.2} \& p_{6.2}) &= \langle \text{point}, 4, \{(\text{blank}, 1), (\text{one}, 2), (\text{zero}, 3), (\text{point}, 4)\}, \{(\text{test}, 3), (\text{carry}, 5), (\text{realign}, 8)\}, \{1, 2, 3, 4\}, 10 = \text{stop}, \{(1, s, 4, 0, 2) \text{ for all } s \in \{1, 2, 3, 4\}, (2, s, s, 0, 5) \text{ for all } s \in \{1, 2, 3, 4\}, (3, s, s, 0, 6) \text{ for all } s \in \{1, 3, 4\}, (3, s, s, 0, 4) \text{ for } s = 2, (4, s, 3, 0, 5) \text{ for all } s \in \{1, 2, 3, 4\}, (5, s, s, -1, 6) \text{ for all } s \in \{1, 2, 3, 4\}, (6, s, s, 0, 3) \text{ for all } s \in \{1, 2, 3, 4\}, (7, s, 2, 0, 8) \text{ for all } s \in \{1, 2, 3, 4\}, (8, s, s, 1, 9) \text{ for all } s \in \{1, 2, 3, 4\}, (9, s, s, 0, 11) \text{ for } s \in \{1, 2, 4\}, (9, s, s, 0, 10) \text{ for } s = 3, (10, s, s, 0, 8) \text{ for all } s \in \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \rangle
\end{aligned}$$

We will get the same values of attributes if we use the evaluation suggested in Knuth system, thus :

$$\begin{aligned}
\Sigma &= \{1, 2, 3, 4\} \\
Q &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \\
\delta &= \{(1, s, 4, 0, 2) \text{ for all } s \in \{1, 2, 3, 4\}, (2, s, s, 0, 5) \text{ for all } s \in \{1, 2, 3, 4\}, (3, s, s, 0, 6) \text{ for all } s \in \{1, 3, 4\}, (3, s, s, 0, 4) \text{ for } s = 2, (4, s, 3, 0, 5) \text{ for all } s \in \{1, 2, 3, 4\}, (5, s, s, -1, 6) \text{ for all } s \in \{1, 2, 3, 4\}, (6, s, s, 0, 3) \text{ for all } s \in \{1, 2, 3, 4\}, (7, s, 2, 0, 8) \text{ for all } s \in \{1, 2, 3, 4\}, (8, s, s, 1, 9) \text{ for } s \in \{1, 2, 4\}, (9, s, s, 0, 10) \text{ for } s = 3, (10, s, s, 0, 8) \text{ for all } s \in \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \}
\end{aligned}$$

all $s \in \{1, 2, 3, 4\}$, $(9, s, s, 0, 11)$ for all $s \in \{1, 2, 4\}$, $(9, s, s, 0, 10)$ for $s = 3$, $(10, s, s, 0, 8)$ for all $s \in \{1, 2, 3, 4\}$

$SY = \{(\text{blank}, 1), (\text{one}, 2), (\text{zero}, 3), (\text{point}, 4)\}$

$L = \{(\text{test}, 3), (\text{carry}, 5), (\text{realign}, 8)\}$

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Note

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University of Bucharest
Faculty of Mathematics

PROFESORUL GHEORGHE GALBURĂ LA 60 DE ANI

Profesorul Gheorghe Galbură, personalitate marcantă a matematicii românești, a împlinit recent 60 de ani. Este o plăcere pentru noi toți, colegi de diferite generații, să sărbătorim cu ocazia acestei aniversări pe unul din cei mai distinși profesori ai Facultății de Matematică a Universității din București.

Gheorghe Galbură s-a născut în Trifești, județul Orhei la 27 mai 1916. După ce a absolvit școala primară în satul natal, a urmat cursurile Școlii Normale din Bacău. O fericită împrejurare a făcut ca talentul său matematic să fie descoperit de profesorul Octav Onicescu care i-a oferit cu generozitate prietenia (o prietenie care se continuă cu aceeași căldură și astăzi) și i-a călăuzit primii pași ca student și tânăr cercetător. Între anii 1936—1939 a urmat cursurile Facultății de Științe a Universității din București, unde a avut ca profesori, printre alții, pe Gheorghe Țițeica și Dan Barbilian. În 1940 a fost numit asistent la Catedra de Algebră a Facultății de Științe din București. La recomandarea lui Octav Onicescu obține o bursă de studii la Roma (între anii 1940 și 1943) la renumitul matematician italian Francesco Severi.

Sub influența lui Severi interesul major al lui Gheorghe Galbură se îndreaptă către geometria algebrică, disciplină pe care o va cultiva în mod constant timp de peste 30 de ani. În 1942 obține la Roma titlul de doctor în matematici cu teza „Sul gruppo caratteristico di una corrispondenza fra varietà algebriche”.

Spiritul geometriei algebrice italiene, aflate pe atunci la apogeul ei, a influențat profund formația matematică a lui Gheorghe Galbură, care a fost fascinant de ideile deosebit de fertile ale unui domeniu matematic ce nu dispunea încă de o fundamentare pe deplin riguroasă.

Mai târziu, pe măsura dezvoltării geometriei algebrice sub impulsul noilor orientări datorate în special lui A. Weil, O. Zariski, J-P. Serre, K. Kodaira și A. Grothendieck, profesorul Galbură a știut să îmbine ingenios avantajele noilor metode cu intuiția specifică geometriei italiene. Ca urmare a activității sale la Facultatea de Matematică, iar după 1949 și la Institutul de Matematică din București, la noi în țară a fost inițiat studiul sistematic al geometriei algebrice, domeniu de mare importanță al matematicii moderne, precum și al unor discipline înrudite ca topologia algebrică și algebra locală, discipline care au fost introduse și în învățământul nostru de specialitate.

Încă din teza sa de doctorat, Gheorghe Galbură s-a arătat interesat în studiul sistemelor canonice atașate unei subvarietăți a unei varietăți algebrice date. Mai târziu, mergând pe linia determinării invariantilor care se atașează unei imersii între două varietăți algebrice, publică o serie de lucrări semnificative dedicate teoriei claselor caracteristice și inelului de echivalență rațională atașat unei varietăți algebrice. A studiat com-

portarea claselor caracteristice la o eclatare a unei varietăți, dînd împreună cu A. Lascu formule explicite în acest sens.

O altă direcție de cercetare abordată cu succes de Gheroghe Galbură a fost aceea a analizei structurii singularităților curbilor algebroidelor plane și suprafețelor algebroidelor în spațiu. Aplicînd metodele și rezultatele obținute, a reușit să dea demonstrații noi și elementare unor formule numerice de mare importanță între diverșii invarianti atașați singularităților curbilor algebrice plane.

Într-o lucrare recentă în colaborare, a pus în evidență echivalența dintre definiția clasică a lui O. Zariski a noțiunii de transformare monoidală a unei varietăți și cea modernă datorată lui A. Grothendieck.

În domeniul topologiei a publicat în 1950 o lucrare importantă în legătură cu varietățile grup de dimensiune 3, iar în 1953, extinzînd un rezultat clasic al lui Riemann, a exprimat genul aritmetic al unei curbe algebrice proiective plane în funcție de gradul curbei și numărul și natura singularităților ei.

A publicat, de asemenea, lucrări de algebră abstractă și algebră locală, monografia „Corpuri de funcții algebrice și varietăți algebrice”, precum și manuale didactice de algebră și geometrie de o deosebită claritate și eleganță a expunerii.

În ultimii 12 ani a condus seminarul de geometrie algebrică organizat de Facultatea de Matematică în colaborare cu Institutul de Matematică al Academiei, reușind să-i imprime o notă dinamică și, pe alocuri, plină de umor spontan.

Activitatea sa ca șef al Catedrei de Algebră și Geometrie a Facultății de Matematică a Universității din București se distinge prin competență, tact și conștiință profesională.

Cei care îl cunosc mai bine apreciază în el un matematician de mare rafinament și un om de o rară modestie și simplitate.

În numele tuturor elevilor săi, încheiem acest modest omagiu, urîndu-i din inimă ani mulți, sănătate și activitate tot atît de rodnică în slujba matematicii românești.

București, 10 iunie 1976

L. Bădescu, M. Moroianu,
Facultatea de Matematică,
Universitatea din București

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